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The Ultimate Cheat Sheet for Astrophysics Students

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Chapter 1

Physics

1.1 Motion

1.1.1 Velocity

- $$\vec{v} = \frac{\Delta\vec{x}}{\Delta t} = \frac{d\vec{x}}{dt} = \dot{\vec{x}}$$

1.1.2 Acceleration

- $$\vec{a} = \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{x}}{dt^2} = \ddot{\vec{x}}$$

1.1.3 Newton's Laws

Newton's First Law

- When viewed in an inertial reference frame, an object either remains at rest or continues to move at a constant velocity, unless acted upon by a net force.

Newton's Second Law

- $$\vec{F}_{net} = m\vec{a} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}$$

Newton's Third Law

- $$\vec{F}_A = -\vec{F}_B$$
- When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.

1.1.4 Momentum

- $$\vec{p} = \gamma m\vec{v} \approx m\vec{v}$$
- $$\Delta\vec{p} = \vec{F}\Delta t$$
- $$\vec{F} = \frac{\Delta\vec{p}}{\Delta t} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}$$

1.1.5 Centripetal Force

- $$F_c = \frac{mv^2}{r}$$

1.1.6 Kinetic Energy

- $$K = \tfrac{1}{2}mv^2$$

1.1.7 Projectile Motion

- $v_y^2 = u_y^2 + 2a_y\Delta y$
- $x = u_x t$
- $\Delta y = u_y \Delta t + \frac{1}{2} a_y \Delta t^2 = u_y t + \frac{1}{2}\dfrac{F_y}{m}\Delta t^2$

1.1.8 Rotation

Angular Velocity

- $\omega = \dfrac{\mathrm{d}\theta}{\mathrm{d}t} = \dot{\theta}$
- $\omega = \dfrac{v}{r}$
- $\vec{v} = \vec{r} \times \vec{\omega}$

Angular Acceleration

- $\alpha = \dfrac{\mathrm{d}\omega}{\mathrm{d}t} = \dfrac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = \dot{\omega} = \ddot{\theta}$

Moment of Inertia

Point Mass

- $I = mr^2$

Several Point Masses

- $I = \sum mr^2$

Continuous mass

- $I = \int r^2 \, \mathrm{d}m$

Parallel axis theorem

- $I = I_{com} + md^2$

Thin disc rotating about centre

- $I = \dfrac{MR^2}{2}$

Thin hoop rotating about centre

- $I = MR^2$

Thin rod rotating about centre

- $I = \dfrac{ML^2}{12}$

Thin rod rotating about end

- $I = \dfrac{ML^2}{3}$

Rotational Kinetic Energy

- $K_{rot} = \frac{1}{2} I\omega^2$

Total Kinetic Energy

- $K_{tot} = K_{trans} + K_{rot} = \frac{1}{2}(mr_{com}^2 + I_{com})\omega^2$

Angular Momentum

- $$\vec{L} = I\vec{\omega} = \vec{r} \times \vec{p}$$

Torque

- $\vec{\tau} = I\vec{\alpha} = \frac{\mathrm{d}L}{\mathrm{d}t} = \vec{r} \times \vec{F}$

1.1.9 Euler-Lagrange and the Hamiltonian

Lagrangian

- $$\ell = T - V = \sum_{lm} a(q)\dot{q}_l\dot{q}_m$$
- $$= K(\dot{q}_l) - U(q_l)$$

Generalised coordinates & momenta

- $$p_k \equiv \frac{\partial L}{\partial \dot{q}_k}$$

Euler-Lagrange Equation

- $$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \ell}{\partial \dot{x}} - \frac{\partial \ell}{\partial x} = 0$$

Action

- $$S[x(t)] = \int_{t_A}^{t_B} \ell(\dot{x}(t), x(t))\, \mathrm{d}t$$

Hamiltonian

- $$\mathcal{H} = \sum_l p_l\dot{q} - L$$
- $$\dot{P} = -\frac{\partial H}{\partial Q}$$
- $$\dot{Q} = \frac{\partial H}{\partial P}$$
- $$\dot{P} = -\omega^2 Q$$
- $$\dot{Q} = P$$

1.2 Oscillations

1.2.1 Springs

Force of a Spring

- $$\vec{F} = -k_s\vec{x}$$

Potential Energy of a Spring

- $$U_s = \frac{1}{2}k_s x^2$$

Angular Frequency of a Spring

- $$\omega = \sqrt{\frac{k_s}{m}}$$

1.3 Materials

1.3.1 Density

- $\rho = \frac{m}{V} = \frac{\mathrm{d}m}{\mathrm{d}V}$

1.4 Energy

1.4.1 Work

- $W = \int_a^b \vec{F} \cdot \mathrm{d}\vec{l} \approx \vec{F} \cdot \vec{s}$

1.5 Forces

1.5.1 Buoyancy (Archimedes' Principle)

- $F_{buoy} = m_{\text{displaced}} g = \rho_d V_d g$

1.5.2 Friction

- $F_K \approx \mu_K F_\perp$
- $F_S \leq \mu_S F_\perp$

1.6 Waves

- $a \sin(\omega t - kx + \phi)$
- $k = \frac{2\pi}{\lambda}$

1.6.1 Wavelength

- $v = f\lambda$

1.6.2 Angular Frequency

- $\omega = \frac{2\pi}{T} = 2\pi f$

1.7 Newtonian Gravity

1.7.1 Force of Gravity

- $\vec{F}_G = \frac{GmM}{r^2}\hat{r} = -m\vec{\nabla}\Phi(\vec{r}) \approx -mg\hat{y} = m\vec{g}$

1.7.2 Gravitational Potential (potential energy per unit mass)

- $\Phi(\vec{r}) = -\sum_i \frac{GM(\vec{r}_i)}{|\vec{r} - \vec{r}_i|} = -\int \frac{G\mu(\vec{r'})}{|\vec{r} - \vec{r'}|}\, \mathrm{d}^3\vec{r'}$

1.7.3 Gravitational field

- $\vec{g}(\vec{r}) = \frac{GM}{r^2} = -\nabla\Phi(\vec{r})$

1.7.4 Gravitational Potential Energy

- $U_G = -\frac{GmM}{r} \approx mgh$

1.7.5 Kepler's Third Law

- $\frac{T^2}{r^3} = \frac{4\pi^2}{G(m+M)} = \text{constant}$

1.8 Electromagnetism

1.8.1 Notation

- $\vec{\mathscr{r}} = \vec{r} - \vec{r'}$

1.8.2 Maxwell's Equations

	Integral form	**Differential form**
Gauss's Law	$\oiint_S \vec{E}\cdot\, \mathrm{d}\vec{a} = \frac{1}{\varepsilon_0}\iiint_V \rho\, \mathrm{d}V$ $= \frac{\sum Q_{\text{enc}}}{\varepsilon_0}$	$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$
Gauss's Law for Magnetism	$\oiint_S \vec{B}\cdot\, \mathrm{d}\vec{a} = 0$	$\vec{\nabla}\cdot\vec{B} = 0$
Maxwell-Faraday equation	$\oint_b \vec{E}\cdot\, \mathrm{d}\vec{l} = -\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \vec{B}\cdot\, \mathrm{d}\vec{a}$	$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$
Ampére's circuital law	$\oint_b \vec{B}\cdot\, \mathrm{d}\vec{l} = \mu_0\iint_S \vec{J}\cdot\, \mathrm{d}\vec{a} + \mu_0\varepsilon_0\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \vec{E}\cdot d\vec{a}$ $= \mu_0(I_{\text{enc}} + \varepsilon_0\frac{\mathrm{d}}{\mathrm{d}t}\int_S \vec{E}\cdot\, \mathrm{d}\vec{a})$	$\vec{\nabla}\times\vec{B} = \mu_0(\vec{J} + \varepsilon_0\frac{\partial\vec{E}}{\partial t})$

1.8.3 Lorentz Force

On a point charge

- $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$

On a current

- $\mathrm{d}\vec{F} = I\int\, \mathrm{d}\vec{l}\times\vec{B}$
- $\vec{F} = \vec{I}L\times\vec{B}$

1.8.4 Electric Field

- $\vec{E} = \int_V \frac{\rho(\vec{r'})}{\mathscr{r}^2}\hat{\mathscr{r}}\, \mathrm{d}\tau$

From a single point charge

- $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$

From a dipole

- $|\vec{E}_axis| \approx \frac{2p}{4\pi\varepsilon_0 r^3}$
- $|\vec{E}_\perp| \approx \frac{p}{4\pi\varepsilon_0 r^3}$

1.8.5 Dipole moment

- $\vec{p} = q\vec{d}$

1.8.6 Electric potential

- $V = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{\mathscr{r}}$
- $\nabla^2 V = \dfrac{-\rho}{\varepsilon_0}$

In a single-point charge field

- $\Delta(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r}$

1.8.7 Electric potential difference

- $\Delta(\vec{r}) = -\int_{\vec{b}}^{\vec{a}} \vec{E} \cdot \, \mathrm{d}\vec{l}$

In a single-point charge field

- $\Delta(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}Q(\dfrac{1}{b} - \dfrac{1}{a})$

1.8.8 Electric potential energy

- $U_E = q\Delta V = \dfrac{1}{4\pi\varepsilon_0}\dfrac{qQ}{\mathscr{r}}$

Energy stored in an electrostatic field distribution

- $U_E = \frac{1}{2} = \epsilon_0 E^2 \times \text{volume}$

1.8.9 Charge densities

Surface

- $\sigma = \dfrac{\mathrm{d}q}{\mathrm{d}a} = \dfrac{Q}{A}$

Line

- $\lambda = \dfrac{\mathrm{d}q}{\mathrm{d}l} = \dfrac{Q}{L}$

1.8.10 Current densities

Volume

- $\vec{J} = \dfrac{\mathrm{d}\vec{I}}{\mathrm{d}\vec{a}_\perp} = \dfrac{I}{A_\perp} = \sigma(\vec{E} + \vec{v} \times B) = |q|nu(\vec{E} + \vec{v} \times B)$
- $\vec{\nabla} \cdot \vec{J} = 0$

Surface

- $\vec{K} = \dfrac{\mathrm{d}\vec{I}}{\mathrm{d}\vec{l}_\perp} = \dfrac{I}{l} = \sigma\vec{v}$

1.8.11 Circuits

Electron drift velocity

- $\bar{v} = u\vec{E}_{\text{net}}$

Current per unit charge

- $i = nA_{cs}\bar{v} = nA_{cs}uE_{\text{net}}$

Current

- $I = ei = enA_{cs}uE_{\text{net}} = \frac{\mathrm{d}q}{\mathrm{d}t}$

Electrical Power

- $P = IV = I^2R$

Voltage (Electric potential difference)

- $V = \Delta V = IR = -\varepsilon$

Electromotive Force (EMF) from a Non-Coulomb force

- $\epsilon = \frac{F_{\text{NC}}d}{e}$

Resistance

- $R = \frac{L\rho}{A} = \frac{L}{\sigma A}$
- $R_{\text{series}} = R_1 + R_2 + ... + R_n$
- $\frac{1}{R_{\text{parallel}}} = \frac{1}{R_1} + \frac{1}{R_2} + ... + \frac{1}{R_n}$

1.8.12 Capacitors

Capacitance

- $C = \frac{Q}{V} = \frac{\varepsilon A}{d} = \frac{k\varepsilon_0 A}{d}$

Energy stored in a capacitor

- $W = \frac{CV^2}{2}$

Electric field in a capacitor

- $E = \frac{Q}{\varepsilon_0 A}$

Potential difference across a capacitor

- $\Delta V = -\frac{dQ}{A\varepsilon_0}$

1.8.13 Magnetic fields

- $\vec{B}(\vec{\mathscr{r}}) = \frac{\mu_0}{4\pi}\int \frac{\vec{I} \times \hat{\mathscr{r}}}{\mathscr{r}^2}\,\mathrm{d}l$
- $\mathrm{d}\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v} \times \hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_0}{4\pi}\frac{I\,\mathrm{d}\vec{l} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$

Magnetic field due to a wire

- $\vec{B} = \frac{\mu_0}{4\pi}\frac{2I}{r}\hat{\phi}$

Magnetic vector potential

- $\vec{A}(\vec{\imath}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r'})}{\imath}\,\mathrm{d}\tau$
- $\vec{\nabla}\times\vec{A} = \vec{B}$
- $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = -\mu_0\vec{J}$
- $\vec{\nabla}\cdot\vec{A} = 0$

1.8.14 Inductors

- $\varepsilon = -LI$

Energy stored in an inductor

- $W = \frac{LI^2}{2}$

1.8.15 Materials

Macroscopic Maxwell's Equations (Materials)

	Integral form	**Differential form**
Gauss's Laws	$\oiint_S \vec{P}\cdot\,\mathrm{d}\vec{a} = -\sum Q_B$ $\oiint_S \vec{D}\cdot\,\mathrm{d}\vec{a} = \sum Q_f$	$\vec{\nabla}\cdot\vec{P} = -\rho_B$ $\vec{\nabla}\cdot\vec{D} = \rho_f$
Gauss's Law for Magnetism	$\oiint_S \vec{B}\cdot\,\mathrm{d}\vec{a} = 0$	$\vec{\nabla}\cdot\vec{B} = 0$
Maxwell-Faraday equation	$\oint_b \vec{E}\cdot\,\mathrm{d}\vec{l} = -\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \vec{B}\cdot\,\mathrm{d}\vec{a}$	$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$
Ampére's circuital law	$\oint_b \vec{H}\cdot\,\mathrm{d}\vec{l} = I_{f,enc} + \frac{\partial}{\partial t}\iint_S \vec{D}\cdot\,\mathrm{d}\vec{a}$	$\vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t}$

Dielectric constant

- $k = \frac{\varepsilon}{\varepsilon_0} = \varepsilon_r$
- $\varepsilon = k\varepsilon_0 = \varepsilon_r\varepsilon$

Susceptibility

- $\chi_e = 1 - \varepsilon_r$

Polarisability

- $\vec{P} = \varepsilon_0\chi_e\vec{E} = n\vec{p}$

Bound Charge

Surface

- $\sigma_B = \vec{p}\cdot\hat{n}$

Volume

- $\rho_B = -\vec{\nabla}\cdot\vec{P}$

Total

- $Q_B = \sigma_B + \rho_B = \vec{p}\cdot\hat{n} - \vec{\nabla}\cdot\vec{P}$

Electric displacement

- $\vec{D} = \varepsilon\vec{E} = k\varepsilon_0\vec{E} = \varepsilon_0\vec{E} + \vec{P}$

Magnetic field

- $\vec{H} = \dfrac{\vec{B}}{\mu_0} - \vec{M}$

Magnetic dipole

- $\vec{m} = I\vec{a}$

Bound current

- $\vec{J}_B = \vec{\nabla} \times \vec{M}$
- $\vec{K}_B = \vec{M} \times \hat{n}$

1.9 Special Relativity

1.9.1 Interval

- $\Delta s^2 = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$
- $\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + \,\mathrm{d}x^2 + \,\mathrm{d}y^2 + \,\mathrm{d}z^2$
- $\Delta s^2 < 0$ is a timelike interval. Events separated by this interval can be causally related.
- $\Delta s^2 = 0$ is a lightlike interval. Events separated by this interval can be causally related, but only by a lightspeed signal.
- $\Delta s^2 > 0$ is a spacelike interval. Events separated by this interval CANNOT be causally related.

Gamma Factor

- $\gamma = \dfrac{1}{\sqrt{1 - (\frac{v}{c})^2}}$
- $\gamma = \dfrac{dt}{d\tau}$

Mass-energy

- $E_{\text{rest}} = mc^2$
- $E = \gamma mc^2 = \dfrac{1}{\sqrt{1 - (\frac{v}{c})^2}} mc^2$

Relativistic kinetic energy

- $K = \gamma mc^2 - mc^2$

Length contraction

- $l_v = \dfrac{l_0}{\gamma} = l_0\sqrt{1 - (\frac{v}{c})^2}$

Time dilation

- $t_v = \gamma t_0 = \dfrac{t_0}{\sqrt{1 - (\frac{v}{c})^2}}$

Mass dilation

- $$m_v = \gamma m_0 = \frac{m_0}{\sqrt{1-(\frac{v}{c})^2}}$$

Relative Velocity

- $$u'_x = \frac{\Delta x'}{\Delta t} = \frac{u_x - v_x}{1 - \frac{v_x u_x}{c^2}}$$

Relativistic Momentum

- $$\vec{p} = \gamma \vec{v} = \frac{m\vec{v}}{\sqrt{1-(v/c)^2}}$$

1.9.2 Four-vectors

Four-space

- $$\mathbf{s} = \mathbf{x} = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

Four-velocity (proper velocity)

- $$\mathbf{u} = \frac{d\mathbf{s}}{d\tau} = \gamma \begin{bmatrix} c \\ v_x \\ v_y \\ v_z \end{bmatrix}$$
- $$\mathbf{u} \cdot \mathbf{u} = -c^2$$

Four-acceleration

- $$\mathbf{w} = \frac{d\mathbf{u}}{d\tau} = \gamma \begin{bmatrix} c \\ v_x \\ v_y \\ v_z \end{bmatrix}$$
- $\mathbf{w} \cdot \mathbf{u} = 0$

Four-momentum

- $$\mathbf{p} = \begin{bmatrix} E/c \\ p_x \\ p_y \\ p_z \end{bmatrix} = \gamma m \begin{bmatrix} c \\ v_x \\ v_y \\ v_z \end{bmatrix} = m\mathbf{u}$$

1.9.3 Frames of Reference

Condition for an inertial frame

- $$\frac{d^2x}{dt^2} = \frac{d^2y}{dt^2} = \frac{d^2z}{dt^2} = 0$$

Galilean Transformations

- $x' = x + vt$
- $y' = y$
- $z' = z$
- All assuming x is along the axis of motion and x = x' when $t = 0$.

Lorentz Boosts

- $t' = \gamma(t - \frac{vx}{c^2})$
- $x' = \gamma(x - vt)$
- $y' = y$
- $z' = z$
- (x is along the axis of motion)
- $\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$

General Lorentz transformation

- $\begin{bmatrix} b'^0 \\ b'^1 \\ b'^2 \\ b'^3 \end{bmatrix} = \begin{bmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b^0 \\ b^1 \\ b^2 \\ b^3 \end{bmatrix}$
- Motion along the x-axis.

Proper Time

- $\tau = \int\limits_{t_A}^{t_B} \frac{1}{\gamma} \, \mathrm{d}t = \int\limits_{t_A}^{t_B} \sqrt{1 - \frac{v^2(t)}{c^2}} \, \mathrm{d}t$

1.10 General Relativity

1.10.1 Metrics

Minkowski

- $\eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{bmatrix}$
- $\mathrm{d}s^2 = -c^2 \, \mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \qquad = -c^2 \, \mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \, \mathrm{d}\theta^2 + r^2 \sin^2\theta \, \mathrm{d}\phi^2$

Schwarzschild

- $g = \begin{bmatrix} -(1 - \frac{2GM}{c^2 r}) & 0 & 0 & 0 \\ 0 & (1 - \frac{2GM}{c^2 r})^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{bmatrix}$
- $\mathrm{d}s^2 = -(1 - \frac{2GM}{c^2 r}) c^2 \, \mathrm{d}t^2 + (1 - \frac{2GM}{c^2 r})^{-1} \, \mathrm{d}r^2 + r^2 \, \mathrm{d}\theta^2 + r^2 \sin^2\theta \, \mathrm{d}\phi^2)$

1.10.2 Rindler coordinates

Line element

- $\mathrm{d}s^2 = -\left(1 + \frac{gx'}{c^2}\right)^2 (c \, \mathrm{d}t')^2 + \mathrm{d}x'$

1.10.3 Einstein summation notation

- $a_\mu b^\mu \equiv \sum_{\mu=0}^{3} a_\mu b^\mu$
- Contravariant: e^α
- Covariant: e_α
- $$t_{\alpha\beta} = g_{\beta\gamma} t_\alpha{}^\gamma$$
- $$t_\alpha{}^\beta = g^{\beta\gamma} t_{\alpha\gamma}$$
- $$t'^\alpha{}_\beta = \frac{\partial x'^\alpha}{\partial x^\gamma}\frac{\partial x^\delta}{\partial x'^\beta} t^\gamma{}_\delta$$
- $$t'_\alpha{}^\beta = \frac{\partial x^\gamma}{\partial x'^\alpha}\frac{\partial x'^\beta}{\partial x^\delta} t_\gamma{}^\delta$$

Metrics

- $$\mathrm{d}s^2 = g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta$$
- $$g^{\alpha\beta} = \frac{1}{g_{\alpha\beta}}$$
- $$\delta^\alpha_\beta = \begin{cases} 1 & \alpha = \beta \\ 1 & \alpha \neq \beta \end{cases}$$
- $$\delta^\alpha_\gamma a^\gamma = a^\alpha$$
- $$g^{\alpha\gamma} g_{\gamma\beta} = \delta^\alpha_\beta$$

Four-vector product

- $$\mathbf{a}\cdot\mathbf{b} = g_{\alpha\beta} a^\alpha b^\beta = a_\beta b^\alpha$$

1.10.4 Christoffel symbols

- $$\Gamma^\alpha{}_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\delta}\left(\frac{\partial g_{\delta\beta}}{\partial x^\gamma} + \frac{\partial g_{\delta\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\delta}\right)$$
- $$\Gamma_{\alpha\beta\gamma} = \tfrac{1}{2}\left(\frac{\partial g_{\delta\beta}}{\partial x^\gamma} + \frac{\partial g_{\delta\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\delta}\right)$$
- $$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau} + \Gamma^\mu{}_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0$$

1.10.5 Covariant derivatives

- $$\nabla_\gamma t^\alpha{}_\beta = \frac{\partial t^\alpha{}_\beta}{\partial x^\gamma} + \Gamma^\alpha{}_{\gamma\delta} t^\delta{}_\beta - \Gamma^\delta{}_{\gamma\beta} t^\alpha{}_\delta$$
- $$\nabla_\gamma t^{\alpha\beta} = \frac{\partial t^{\alpha\beta}}{\partial x^\gamma} + \Gamma^\alpha{}_{\gamma\delta} t^{\delta\beta} + \Gamma^\beta{}_{\gamma\delta} t^{\alpha\delta}$$
- $$\nabla_\gamma t_{\alpha\beta} = \frac{\partial t_{\alpha\beta}}{\partial x^\gamma} - \Gamma^\delta{}_{\gamma\alpha} t_{\delta\beta} - \Gamma^\delta{}_{\gamma\beta} t^{\alpha\delta}$$
- $$\nabla_\gamma t_\alpha{}^\beta = \frac{\partial t_\alpha{}^\beta}{\partial x^\gamma} - \Gamma^\delta{}_{\gamma\alpha} t_\delta{}^\beta + \Gamma^\beta{}_{\gamma\delta} t_\alpha{}^\delta$$

1.10.6 Riemann curvature tensor

- $R^{\alpha}{}_{\beta\gamma\delta} = \dfrac{\partial \Gamma^{\alpha}{}_{\beta\delta}}{\partial x^{\gamma}} - \dfrac{\partial \Gamma^{\alpha}{}_{\beta\gamma}}{\partial x^{\delta}} + \Gamma^{\alpha}{}_{\gamma\epsilon}\Gamma^{\epsilon}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\delta\epsilon}\Gamma^{\epsilon}{}_{\beta\gamma}$
- $R_{\alpha\beta\gamma\delta} = \frac{1}{2}(\dfrac{\partial^2 g_{\alpha\delta}}{\partial x^{\beta}\partial x^{\gamma}} - \dfrac{\partial^2 g_{\alpha\gamma}}{\partial x^{\beta}\partial x^{\delta}} - \dfrac{\partial^2 g_{\beta\delta}}{\partial x^{\alpha}\partial x^{\gamma}}) + \dfrac{\partial^2 g_{\beta\gamma}}{\partial x^{\alpha}\partial x^{\delta}}$
- $R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}$
- $R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\delta\gamma}$
- $R_{\alpha\beta\gamma\delta} = R_{\delta\gamma\alpha\beta}$
- $R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta=0}$

1.10.7 Ricci curvature tensor

- $R_{\alpha\beta} = R^{\gamma}{}_{\alpha\gamma\beta}$
- $R = R^{\alpha}{}_{\alpha}$

1.10.8 Einstein's equations

- $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \dfrac{8\pi G}{c^4}T_{\mu\nu}$

1.11 Thermodynamics

1.11.1 Ideal Gases

Ideal Gas Law

- $pV = Nk_BT$

Heat / Thermal Energy

- $Q = mc\Delta T$

Heat Capacity

- $C = \dfrac{\mathrm{d}Q}{\mathrm{d}T}$

Specific Heat Capacity

- $c = \dfrac{C}{m}$

1.11.2 Microstates

- $\Omega = \dfrac{(q + N - 1)}{q!(N-1)}$

1.11.3 Entropy

- $S = k_B \ln \Omega$

1.11.4 Black bodies

Energy of a photon

- $E = hf$

Wien's Displacement Law

- $\lambda_{max} = \dfrac{b}{T} = (2.8977729 \times 10^{-3})\dfrac{1}{T}$

Stefan-Boltzmann Law

- $$I = \sigma T^4$$

Spectrum

- $$B_\lambda(T) = \frac{2hc^2}{\lambda^5}\frac{1}{\exp(\frac{hc}{\lambda k_B T} - 1)}$$
- $$B_\nu(T) = \frac{2h\nu}{c^2}\frac{1}{\exp(\frac{h\nu}{k_B T}) - 1}$$

1.12 Quantum Mechanics

1.12.1 The Uncertainty Principle

- $\Delta x \Delta p \geq \frac{\hbar}{2}$
- $\Delta E \Delta t \geq \frac{\hbar}{2}$

1.12.2 Bras and Kets

- $$|\psi\rangle = \langle\psi|^\dagger$$

1.12.3 Rules for an Inner Product

- $$\langle\psi|\phi\rangle \equiv (|\psi\rangle, |\phi\rangle)$$
- Symmetric:
  $$\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$$
- Linear in second component
- Anti-linear in first component

1.12.4 The Born Rule

- $$P = |\langle\psi|\psi\rangle|^2$$

1.12.5 Expectation

- $$\langle A\rangle = \int A|\Psi(x,t)|^2\,\mathrm{d}x$$
- $$\langle A\rangle = \langle\psi|A|\psi\rangle$$

1.12.6 Variance

- $$\mathrm{var}(A) = \langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2$$

1.12.7 Standard Deviation

- $$\delta A = \sqrt{\mathrm{var}(A)} = \sqrt{\langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2}$$

1.12.8 Trace

- $$\mathrm{Tr}(A) = \sum_j \langle x_j|A|x_j\rangle$$

1.12.9 Partial Trace

- $\text{Tr}_B(|a\rangle\langle a| \otimes |b\rangle\langle b|) \equiv |a\rangle\langle a|\text{Tr}(|b\rangle\langle b|)$
- $\text{Tr}(k_{AB}) = \text{Tr}_A(\text{Tr}_B(k_{AB})) = \text{Tr}_B(Tr_A(k_{AB}))$
- $\rho_B = \text{Tr}_A(\rho_{AB})$
- The partial trace is linear

1.12.10 The Schrödinger Equation

- $i\hbar\frac{\partial}{\partial t}\Psi(r,t) = \hat{H}\Psi(r,t)$
- $-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x)\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t}$
- $-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + V(x)\psi(x,t) = E\psi(x)$
- $\hat{H}|\Psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle$

1.12.11 Heisenberg equation of motion

- $\frac{d}{dt}\hat{A}(t) = \frac{i}{\hbar}[\hat{H}, \hat{A}(t)]$

1.12.12 Operators

- $a_{jk} = \langle j|A|k\rangle$

Diagonalizable Operator

- $A = \sum_j \lambda_j|\lambda_j\rangle\langle\lambda_j|$

Normal Operator

- $A = \sum_j|\lambda_j\rangle\langle\lambda_j|$

Eigenstate Operators

- $(|\lambda_k\rangle\langle\lambda_k|)^n = |\lambda_k\rangle\langle\lambda_k|$

Identity

- $I = \sum_j|x_j\rangle\langle x_j|$

Projector

- $P = |\psi\rangle\langle\psi|$

Density operator

- $\rho \equiv \sum_j P_j|\psi_j\rangle\langle\psi_j|$
- Hermitian: $\rho^\dagger = \rho$
- Normalised: $\text{Tr}(\rho) = 1$
- Positive Semi-Definite: $\langle\psi|\rho|\psi\rangle \geq 0, \ \forall \ |\psi\rangle \in \mathbf{H}$
- $purity = \text{Tr}(\rho^2)$

- $\frac{1}{d} \leq \mathrm{Tr}(\rho^2) \leq 1$
- Pure: $\mathrm{Tr}(\rho^2) = 1$
- Maximally mixed: $\mathrm{Tr}(\rho^2) = \frac{1}{d}$

- $\rho_A = \mathrm{Tr}_B(\rho_{AB})$
- $\langle A \rangle = \mathrm{Tr}(\rho A)$

Pauli Operators

- $\sigma_x = X = |0\rangle\langle 1| + |1\rangle\langle 0| \doteq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- $\sigma_y = Y = i|1\rangle\langle 0| - i|0\rangle\langle 1| \doteq \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
- $\sigma_z = Z = |0\rangle\langle 0| - |1\rangle\langle 1| \doteq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- $I = |0\rangle\langle 0| + |1\rangle\langle 1| \doteq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (Sometimes included)
- $\mathrm{Tr}(X) = \mathrm{Tr}(Y) = \mathrm{Tr}(Z) = 0$
- With respect to Hilbert-Schmidt Inner Product:
  $||X|| = ||Y|| = ||Z|| = ||I|| = \sqrt{2}$

Properties

- Unitary
- Hermitian
- $\lambda = \pm 1$

Photon Annihilation and Creation Operators

- $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$
- $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$
- $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$
- $\langle\alpha|\hat{a}^\dagger = \alpha^*\langle\alpha|$

Atomic Energy Level Operators (for a two-level approximation)

- $\hat{\sigma}_+ = |e\rangle\langle g|$
- $\hat{\sigma}_- = |g\rangle\langle e|$
- $\hat{\sigma}_z = |e\rangle\langle e| - |g\rangle\langle g|$
- $\hat{\sigma}_+|g\rangle = |e\rangle$
- $\hat{\sigma}_-|e\rangle = |g\rangle$
- $\hat{\sigma}_+|e\rangle = 0$
- $\hat{\sigma}_-|g\rangle = 0$

Chapter 2

Astrophysics & Astronomy

2.1 Astrometry

2.1.1 Redshift

- $$z = \frac{\lambda_{\text{obs}} - \lambda_{\text{emit}}}{\lambda_{\text{emit}}} = \frac{f_{\text{emit}} - f_{\text{obs}}}{f_{\text{obs}}} \approx \frac{v}{c} \qquad (v << c)$$
- $$1 + z = \frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}} = \frac{f_{\text{emit}}}{f_{\text{obs}}}$$

2.1.2 Apparent magnitude

- $$m - m_0 = -2.5 \log_{10}(\frac{F}{F_0})$$

2.1.3 Absolute magnitude

- $$M = m - 5 \log_{10}(\frac{d}{10})$$

2.1.4 Relative magnitudes

- $$\frac{I_a}{I_b} = 100^{\frac{(m_b - m_a)}{5}}$$

2.1.5 Flux-magnitude relationship

- $$F = F_0 \times 10^{-0.4m}$$

2.1.6 Color

- $$-2.5 \log(\frac{F_{f1}}{F_{f2}})$$

2.1.7 Metallicity

- $$Z = \log_{10}(\frac{(Fe/H)_*}{(Fe/H)_\odot}) = \log_{10}(Fe/H)_* - log_{10}(Fe/H)_\odot$$

2.2 Stars

2.2.1 Stellar Structure Equations

Hydrostatic Equilibrium

- $$\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{-GM_r\rho(r)}{r^2}$$

Mass Conservation

- $$\frac{M_r}{r} = 4\pi r^2 \rho$$

Energy Equation

- $$\frac{\mathrm{d}L_r}{\mathrm{d}r} = 4\pi r^2 \rho \varepsilon$$

Radiative Transport

- $$\frac{\mathrm{d}T}{\mathrm{d}r}|_{rad} = \frac{3}{4ac} \frac{\bar{\kappa}\rho}{T^3} \frac{L_r}{4\pi r^2}$$

2.2.2 Luminosity

- $L = 4\pi R^2 \sigma T_{eff}^4$

2.2.3 Timescales

Thermal / Kelvin-Helmholtz Timescale

- $$\tau_{KH} = \frac{|U_*|}{L_*} = \frac{GM_*^2}{R_* L_*}$$
- $\tau_{KH\odot} \approx$ 50 million years

Dynamical Timescale

- $$\tau_{dyn} \approx \sqrt{\frac{R^3}{GM}} \approx \sqrt{G\bar{\rho}}$$

Nuclear Timescale / Main Sequence Lifespan

- $$\tau_N \approx \tau_\odot M^{-3} \approx 10^9 (\frac{M}{M_\odot})^{-3}$$

2.2.4 Gravitational potential energy

- $$U_* \approx \frac{-GM^2}{R}$$

2.2.5 Eddington Limit (hydrostatic equilibrium)

- $$L_{edd} = \frac{4\pi G M m_p c}{\sigma T} \approx 3.2 \times 10^4 (\frac{M}{M_\odot})\ [L_\odot]$$
- $$M_{edd} = 3.1 * 10^{-5} (\frac{L}{L_\odot})\ [M_\odot]$$

Eddington Rate (mass loss)

- $$\dot{M}_{edd} = \frac{L_{edd}}{\eta c^2} \approx 2.4 \times 10^{-8} (\frac{M}{M_\odot})\ [M_\odot/yr]$$

2.2.6 Mass-Luminosity Relationship

- $$\frac{L}{L_\odot} \approx b \left(\frac{M}{M_\odot}\right)^a \; ; \qquad a, b = \begin{cases} 2.3,\ 0.23 & M < 0.43 M_\odot \\ 4,\ 1 & 0.43 M_\odot < M < 2 M_\odot \\ 3.5,\ 1.5 & 2 M_\odot < M < 20 M_\odot \\ 1,\ 32000 & M > 55 M_\odot \end{cases}$$

2.3 Galaxies

2.3.1 Hubble Eliptical Galaxy Classification

- $$10 \times \left(\frac{a-b}{a}\right)$$

2.3.2 Sérsic Profile

- $I(R) = I_0 \exp\{-b[\left(\frac{R}{R_e}\right)^{\frac{1}{n}} - 1]\}$

2.3.3 Density of stars in the Milky Way Galaxy

- $\rho(R, z) = \rho_0 e^{-z/z_0} e^{-R/h}$

2.4 Black Holes

2.4.1 Schwarszchild Radius

- $$r_S = \frac{2GM}{c^2}$$

2.5 Instrumentation

2.5.1 Lensmaker's equation

- $$\begin{aligned} \frac{1}{f} &= (n-1)[\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n-1)d}{nR_1R_2}] \\ &\approx (n-1)[\frac{1}{R_1} - \frac{1}{R_2}] \qquad \text{(Thin lens approximation)} \end{aligned}$$

2.5.2 Focal ratio / Focal number

- $$N = \frac{f}{D}$$

2.5.3 Field of view

- $$FOV = \frac{w_D}{D_T N} = \frac{w_D}{f_{\text{sys}}}$$

2.5.4 Resolution Limits

Diffraction limit (Rayleigh criterion)

- $$\varepsilon_d = 1.22\frac{\lambda}{D}$$

Seeing limit (Rayleigh criterion)

- $$\varepsilon_s = 0.98\frac{\lambda}{r_0}$$

Total Resolution limit

- $$\varepsilon\sqrt{\varepsilon_d^2 + \varepsilon_s^2}$$

2.5.5 Nyquist sampling

- $$\frac{2p}{f_{\text{sys}}} = \frac{\lambda}{D_T} \qquad \text{(When diffraction limited)}$$
- $$N = \frac{2p}{\lambda}$$

2.5.6 Plate scale

- $$\frac{1}{f}[\text{rad}/m] = \frac{206265}{f}[\text{arcsec}/m]$$

2.5.7 Fitting error

- $$\sigma^2_{\text{fit}} = a_f(\frac{d_{sub}}{r_0})^{\frac{5}{3}} = 0.26(\frac{d_{sub}}{r_0})^{\frac{5}{3}}$$

2.5.8 Adaptive optics error

- $$\sigma^2_{\text{total}} = 0.3(\frac{d_{\text{sub}}}{r_0})^{\frac{5}{3}} + (\frac{\theta}{\theta_0})^{\frac{5}{3}} + 28.4(\frac{\tau}{\tau_0})^{\frac{5}{3}} + C_{WFS}(\frac{\lambda}{F\tau d_{\text{sub}}})^2$$

2.5.9 Signal-to-noise ratio

- $$\text{SNR} = \frac{Ft}{\sqrt{Ft + (B_s n_p t) + (D n_p t) + (R^2 n_p)}}$$

2.5.10 Atmospheric Extinction

- $$m_\lambda = m_{\lambda,z} - a_\lambda(\sec z)$$

2.5.11 Rocket science

Tsiolkovsky rocket equation

- $$\Delta v = v_e \ln(\frac{m_0}{m_f})$$

Chapter 3

Mathematics

3.1 Notation

- $[f(x)]_b^a = f(a) - f(b)$

3.2 Algebra

3.2.1 Factorisation

- $(a+b)^2 = a^2 + b^2 + 2ab^2$
- $(a-b)^2 = a^2 + b^2 - 2ab^2$
- $a^2 - b^2 = (a+b)(a-b)$
- $(a+b)(a+c) = a^2 + (b+c)a + bc$
- $(a+b)^3 = a^3 + 3ab^2 + 3a^2b + b^3$
- $(a-b)^3 = a^3 + 3ab^2 - 3a^2b - b^3$
- $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
- $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
- $a^{2n} - b^{2n} = (a^n - b^n)(a^n + b^n)$

3.2.2 Absolute Value

- $|ab| = |a||b|$
- $|\frac{a}{b}| = \frac{|a|}{|b|}$
- $|a+b| \leq |a| + |b|$

3.2.3 Quadratics

Quadratic Formula

For $ax^2 + bx + c = 0$, $a \neq 0$:

- x = $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- $b^2 - 4ac > 0$ - two real unequal solutions.
- $b^2 - 4ac = 0$ - repeated real solution.
- $b^2 - 4ac < 0$ - two complex solutions.

3.2.4 Logarithms

- $y = \log_b(x);\ x = b^y$
- $\log_b(xy) = \log_b x + \log_b(y)$
- $\log_b(\frac{x}{y}) = \log_b x - \log_b(y)$
- $\log_b(x^p) = p \log_b x$
- $\log_b(b^x) = x$
- $\log_b(a) = \frac{\log_d(a)}{\log_d(b)}$
- $\log_b(\sqrt[p]{x}) = \log$
- $p \log_b x + q \log_b(y) = \log(x^p y^q)$
- $b^{\log_b x} = x$
- $\log_b(b) = 1$
- $\log_b(1) = 0$

3.2.5 Vectors

Dot Product

- $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n = ab\cos\theta$

Cross Product

- $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\hat{i}\ + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k} = ab\sin\theta\hat{n}$
- $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
- $\vec{a} \times \vec{b} = 0$

Vector Product Identities

- $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b}))$

3.2.6 Factorials

- $n! = n(n-1)(n-2)...(2)(1)$
- $(n+1)! = (n+1)n!$

Stirling's approximation

- $n! \approx n\ln(n) - n + O(\ln(n))$

The Factorial Integral

- $\int\limits_0^\infty x^n e^{-x}\,\mathrm{d}x$

3.2.7 Inner product definition

1. Linear in first variable:

   $$(\alpha a + \beta b, c) = \alpha(a, c) + \beta(b, c)$$

2. Positive-definite:

   $$(a, a) \geq 0, (a, a) = 0 \iff a = 0$$

3. Conjugate symmetrical:

   $$(a, b) = (b, a)^*$$
   $$(a, b) = (b, a),\ b, a \in \mathbf{R}$$

3.2.8 Complex Numbers

- $$z = a + ib = \Re(z) + \Im(z)i$$

Euler's Formula

- $$e^{i\theta} = \cos\theta + i\sin\theta$$
- $$re^{i\theta} = |z|e^{i\theta} = r(\cos\theta + i\sin\theta)$$

De Moivre's Formula

- $$(\cos x + i\sin x)^n = \cos(nx) + i\sin(nx)$$

Complex Modulus

- $$r = |z| = |a + ib| = \sqrt{a^2 + b^2} = \sqrt{\Re^2(z) + \Im^2(z)}$$

Complex Conjugate

- $$\bar{z} = (\bar{a + ib}) = a - ib$$
- $$(\bar{a + ib})(a + ib) = |a + ib|^2$$

Complex Argument

- $$\theta = \arg(z) = \arctan(\frac{a}{b}) = \arctan(\frac{\Im(z)}{\Re(z)})$$

3.2.9 Power Series

- $$f(x) = \sum_{n=0}^{\infty} f_n x^n$$

Notable Series

- $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \dots$$
- $$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$$
- $$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...$$

3.2.10 Matrix Operations

Determinant

- $$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Transpose

- $a_{jk}^T = a_{kj}$
- $(AB)^T = B^T A^T$
- Linear: $(A + B)^T = A^T + B^T$
- $(rA)^T = rA^T$

Hermitian Adjoint

- $A^\dagger \equiv (A^*)^T = (A^T)^*$
- $(AB)^\dagger = B^\dagger A^\dagger$
- $(A + B)^\dagger = A^\dagger + B^\dagger$

Trace

- $\text{Tr}(A) = \sum_j^n a_{jj}$
- Cyclic: $\text{Tr}(AB) = \text{Tr}(AB)$
- Linear: $\text{Tr}(A + B) = \text{Tr}(A) + \text{Tr}(B)$
  $\text{Tr}(aB) = a\text{Tr}(B), a \in \mathbf{C}$
- $\text{Tr}(SAS^{-1}) = \text{Tr}(A)$

Hilbert-Schmidt Inner Product

- $(A, B) \equiv \text{Tr}(A^\dagger B)$

Rank-Nullity Theorem

- $\text{rk}(A) = \dim(\ker(A)) + \dim(\text{im}(A))$

3.2.11 Matrix Types

Real Matrix

- $a_{jk} \in \mathbf{R}$

Square Matrix

- $m = n$

Symmetric Matrix

- $A = A^T$
- $a_{jk} = a_{kj}$
- Square

Normal Matrix

- $A^\dagger A = AA^\dagger$
- Square
- Diagonalisable

Diagonal Matrix

- $a_{jk} = 0, j \neq k$
- $\lambda_j = a_{jj}$
- Square
- $e^D = \begin{bmatrix} e^{d_{11}} & 0 & \cdots & 0 \\ 0 & e^{d_{11}} & \cdots & 0 \end{bmatrix}$

Diagonalisable Matrix

- $A = PDP^{-1}$
- Square

Identity Matrix

- $IA = A$
- $i_{jj} = 1$
- $i_{jk} = 0, j \neq k$
- Real
- Square
- Diagonal
- Symmetric
- Hermitian

Hermitian Matrix

- $H = H^\dagger$
- $h_{jk} = h^*_{kj}$
- $h_{jj} \in \mathbf{R}$
- $\lambda \in \mathbf{R}$
- Square
- Normal
- All real, square matrices are Hermitian

Anti-Hermitian Matrix

- $H = -H^\dagger$
- $H_{jk} = -H^*_{kj}$
- Square

Orthogonal Matrix

- $A^T = A^{-1}$
- $AA^T = I$
- $(AA^T)_{jk} = \delta_{jk}$

Positive Semidefinite

- $A \geq 0$
- $\hat{A}^{\dagger} = \hat{A}, \hat{A} \geq 0$
- $B = \hat{A}^{\dagger}\hat{A}$ is positive semidefinite for any linear operator $\hat{A}$
- Positive semidefinite matrices are Hermitian

Projector

- $\hat{P}^2 = \hat{P}$
- $\lambda = 1 \ or \ 0$
- $P_1P_2 \mapsto \mathbf{H}_1 \cap \mathbf{H}_2$
- Projectors are Hermitian

Real Matrix

- $A = A^*$
- $A_{jk} = A^*_{jk}$

Imaginary Matrix

- $A = -A^*$
- $A_{jk} = -A^*_{jk}$

Symmetric Matrix

- $A = A^T$
- $A_{jk} = A_{kj}$
- Square

Antisymmetric Matrix

- $A = -A^T$
- $a_{jk} = a_{kj}$
- $a_{jj} = 0$
- Square

Unitary Matrix

- $U^{\dagger}U = UU^{\dagger} = I$
- $U^{\dagger} = U^{-1}$
- $(U^{\dagger}U)_{jk} = \delta_{jk}$
- Square
- Normal
- Hermitian

3.2.12 Change of Basis Unitary

- $(V)_b = [U^{\dagger}]_a(V)_a$
- $[U]_a = [(b_0)_a(b_1)_a...(b_n)_a]$

3.2.13 Commutator

- $[A, B] = AB - BA$
- $[A, A] = 0$
- $[A + B, C] = [A, C] + [B, C]$
- $[A, BC] = [A, B]C + B[A, C]$

3.2.14 Anticommutator

- $\{A, B\} = AB + BA$

3.2.15 Cauchy-Schwarz Inequality

- $|\langle \vec{u}, \vec{v} \rangle|^2 \leq \langle \vec{u}, \vec{u} \rangle \cdot \langle \vec{v}, \vec{v} \rangle$

3.3 Geometry

3.3.1 Pythagorean theorem

- $a^2 + b^2 = c^2$
- $a = \sqrt{b^2 + c^2}$

Higher dimensions

- $r = \sqrt{x^2 + y^2 + z^2}$
- $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\vec{r} \cdot \vec{r}}$

Distance between two points

In two dimensions

- $d_{12} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

higher dimensions

- $d_{ab} = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + \cdots + (b_n - a_n)^2}$

3.3.2 Properties of shapes

	Area	**Circumference**
Circle	πR^2	$2\pi R$
Square	L^2	$4L$

3.3.3 Properties of solids

	Surface Area	**Volume**
Sphere	$4\pi R^2$	$\frac{4}{3}\pi R^3$

3.3.4 Circular formulae

Arc length

- $l = R\theta$

Area of a sector

- $A = \dfrac{R^2\theta}{2}$

Area of a segment

- $$A = \frac{R^2}{2}(\theta - \sin\theta)$$

3.3.5 Useful Functions

Parabola

- $f(x) = a(x-h)^2 + k$
- Vertex at (h, k)
- Up-concave if $a > 0$; down-concave if $a < 0$

- $f(x) = ax^2 + bx + c$
- Vertex at $(-\frac{b}{2a}, f(-\frac{b}{2a}))$
- Up-concave if $a > 0$; down-concave if $a < 0$

Hyperbola

- $(\frac{x-h}{a})^2 - (\frac{y-k}{b})^2 = 1$
- Centre at (h, k)
- Asymptotes through centre, slope $\pm\frac{b}{a}$

Circle

- $(x-h)^2 + (y-k)^2 = R^2$
- Centre at (h, k)

Ellipse

- $1 = (\frac{x-h}{a})^2 + (\frac{y-k}{b})^2$
- Centre at (h, k)
- Vertices a units right/left from the centre and vertices b units up/down from the center.

Sphere

- $R^2 = (x-h)^2 + (y-k)^2 + (z-l)^2$
- Centre at (h, k, l):

Ball

- $R^2 < (x-h)^2 + (y-k)^2 + (z-l)^2$
- Centre at (h, k, l):

3.3.6 Coordinates

Transformations to Cartesian coordinates

Cartesian	$x = x$	$y = y$	$z = z$	$dV = dx\, dy\, dz$
Polar (2D)	$x = r\cos\phi$	$y = r\sin\phi$	N/A	$dA = r\, drd\phi$
Cylindrical	$x = r\cos\phi$	$y = r\sin\phi$	$z = z$	$dV = r\, dr\, d\theta\, dz$
Spherical	$x = r\sin\theta\cos\phi$	$y = r\sin\theta\sin\phi$	$z = r\cos\theta$	$dV = r^2\sin\theta\, dr\, d\theta\, d\phi$

Transformations from Cartesian coordinates

Cartesian	$x = x$	$y = y$	$z = z$
Polar (2D)	$r = \sqrt{x^2 + y^2}$	$\phi' = \arctan\lvert\frac{y}{x}\rvert$ (ϕ depends on quadrant)	N/A
Cylindrical	$r = \sqrt{x^2 + y^2}$	$\phi' = \arctan\lvert\frac{y}{x}\rvert$ (ϕ depends on quadrant)	$z = z$
Spherical	$r = \sqrt{x^2 + y^2 + z^2}$	$\phi' = \arctan\lvert\frac{y}{x}\rvert$ (ϕ depends on quadrant)	$\theta = \arccos(\frac{z}{\sqrt{x^2 + y^2 + z^2}})$

3.3.7 Hyperbolic Functions

Hyberbolic Sine

- $$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2e^x} = \frac{1 - e^{-2x}}{2e^{-x}}$$
- $\sinh x = -i\sin(ix)$

Hyperbolic Cosine

- $$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2e^x} = \frac{1 + e^{-2x}}{2e^{-x}}$$
- $\cosh x = \cos(ix)$

Hyperbolic Tangent

- $$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{1 - e^{-2x}}{1 + e^{-2x}}$$
- $\tanh x = -i\tan(ix)$

Hyperbolic Cotangent

- $$\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x} + 1}{e^{2x} - 1} = \frac{1 + e^{-2x}}{1 - e^{-2x}}$$
- $\coth x = i\cot(ix)$

Hyperbolic Secant

- $$\mathrm{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} = \frac{2e^x}{e^{2x} + 1} = \frac{2e^{-x}}{1 + e^{-2x}}$$
- $\mathrm{sech} x = \sec(ix)$

Hyperbolic Cosecant

- $\text{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} = \frac{2e^x}{e^{2x} - 1} = \frac{2e^{-x}}{1 - e^{-2x}}$
- $\text{csch} x = i \csc(ix)$

Identities

- $\cosh^2 x - \sinh^2 x = 1$
- $\sin\theta \cos\theta = \frac{1}{2}\sin(2\theta)$

3.4 Trigonometry

Definitions

- SOH CAH TOA
- $\sin\theta = \frac{opposite}{hypotenuse}$
- $\cos\theta = \frac{adjacent}{hypotenuse}$
- $\tan\theta = \frac{opposite}{adjacent} = \frac{\sin\theta}{\cos\theta}$
- $\cot\theta = \frac{adjacent}{opposite} = \frac{\cos\theta}{\sin\theta}$
- $\sec\theta = \frac{hypotenuse}{adjacent}$
- $\csc\theta = \frac{hypotenuse}{opposite}$

3.4.1 Identities

Pythagorean Identities

- $\cos^2\theta + \sin^2\theta = 1$
- $\tan^2\theta + 1 \sec^2\theta$
- $1 + \cot^2\theta = \csc^2\theta$

Reciprocals

- $\sin\theta = \frac{1}{\csc\theta}$
- $\cos\theta = \frac{1}{\sec\theta}$
- $\tan\theta = \frac{1}{\cot\theta}$
- $\cot\theta = \frac{1}{\tan\theta}$
- $\sec\theta = \frac{1}{\cos\theta}$
- $\csc\theta = \frac{1}{\sin\theta}$

As complex exponentials

- $\sin\theta = \dfrac{e^{ix} - e^{-ix}}{2i}$
- $\cos\theta = \dfrac{e^{ix} + e^{-ix}}{2}$
- $\tan\theta = \dfrac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})}$
- $\cot\theta = \dfrac{i(e^{ix} + e^{-ix})}{e^{ix} - e^{-ix}}$
- $\sec\theta = \dfrac{2}{e^{ix} + e^{-ix}}$
- $\csc\theta = \dfrac{2i}{e^{ix} - e^{-ix}}$

Symmetries

- $\sin(-\theta) = -\sin\theta$
- $\cos(-\theta) = \cos\theta$
- $\tan(-\theta) = -\tan\theta$
- $\csc(-\theta) = -\csc\theta$
- $\sec(-\theta) = \sec\theta$
- $\cot(-\theta) = -\cot\theta$

- $\sin(\frac{\pi}{2} - \theta) = \cos\theta$
- $\cos(\frac{\pi}{2} - \theta) = \sin\theta$
- $\tan(\frac{\pi}{2} - \theta) = \cot\theta$
- $\csc(\frac{\pi}{2} - \theta) = \sec\theta$
- $\sec(\frac{\pi}{2} - \theta) = \csc\theta$
- $\cot(\frac{\pi}{2} - \theta) = \tan\theta$

- $\sin(\pi - \theta) = \sin\theta$
- $\cos(\pi - \theta) = -\cos\theta$
- $\tan(\pi - \theta) = -\tan\theta$
- $\csc(\pi - \theta) = \csc\theta$
- $\sec(\pi - \theta) = -\sec\theta$
- $\cot(\pi - \theta) = -\cot\theta$

Angle sum and difference formulae

- $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
- $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
- $\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

Half-angle formulae

- $\sin^2(\frac{\theta}{2}) = \frac{1-\cos\theta}{2}$
- $\cos^2(\frac{\theta}{2}) = \frac{1+\cos\theta}{2}$
- $\tan^2(\frac{\theta}{2}) = \frac{1-\cos\theta}{1+\cos\theta}$
- $\tan(\frac{\theta}{2}) = \frac{\tan\theta}{1+\sec\theta} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta} = \csc\theta - \cot\theta$

Double-angle formulae

- $\cos(2\theta) = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \cos^2\theta - \sin^2\theta$
- $\sin(2\theta) = 2\sin\theta\cos\theta$
- $\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$

Sum to Product

- $\sin\alpha + \sin\beta = 2\sin(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$
- $\sin\alpha - \sin\beta = 2\cos(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$
- $\cos\alpha + \cos\beta = 2\cos(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$
- $\cos\alpha - \cos\beta = -2\sin(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$

Product to Sum

- $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$
- $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$
- $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \cos(\alpha-\beta)]$
- $\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$

Law of Sines

- $\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$

Law of Cosines

- $a^2 = b^2 + c^2 - 2bc\cos\alpha$
- $b^2 = a^2 + c^2 - 2ac\cos\beta$
- $c^2 = a^2 + b^2 - 2ab\cos\gamma$

Law of Tangents

- $\frac{a-b}{a+b} = \frac{\tan(\frac{1}{2}[\alpha-\beta])}{\tan(\frac{1}{2}[\alpha+\beta])}$
- $\frac{b-c}{b+c} = \frac{\tan(\frac{1}{2}[\beta-\gamma])}{\tan(\frac{1}{2}[\beta+\gamma])}$
- $\frac{a-c}{a+c} = \frac{\tan(\frac{1}{2}[\alpha-\gamma])}{\tan(\frac{1}{2}[\alpha+\gamma])}$

Mollweide's Formula

- $$\frac{a+b}{c} = \frac{\cos(\frac{1}{2}[\alpha-\beta])}{\sin(\frac{1}{2}\gamma)}$$

Small-angle approximations

- $\sin\theta \approx \theta$
- $\cos\theta \approx 1 - \frac{\theta^2}{2}$
- $\tan\theta \approx \theta$

Other identities

- $\sin\theta\cos\theta = \frac{1}{2}\sin(2\theta)$
- $\cos^2\theta = \frac{1}{2}(\cos(2\theta)+1)$

Averages

- $\bar{\sin}\, x = \bar{\cos}\, x = 0$
- $\bar{\sin^2}\, x = \bar{\cos^2}\, x = \frac{1}{2}$

Table of Identities

In terms of...	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\sec\theta$	$\cot\theta$	$\csc\theta$
$\sin\theta =$	$\sin\theta$	$\pm\sqrt{1-\cos^2\theta}$	$\pm\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$	$\pm\frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\pm\frac{1}{\sqrt{1+\cot^2\theta}}$	$\frac{1}{\csc}$
$\cos\theta =$	$\pm\sqrt{1-\sin^2\theta}$	$\cos\theta$	$\pm\frac{1}{\sqrt{1+\tan^2\theta}}$	$\frac{1}{\sec\theta}$	$\pm\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$	$\pm\frac{\sqrt{\csc^2-1}}{\csc\theta}$
$\tan\theta =$	$\pm\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\pm\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	$\tan\theta$	$\pm\sqrt{\sec^2\theta-1}$	$\frac{1}{\cot\theta}$	$\pm\frac{1}{\sqrt{\csc^2\theta-1}}$
$\sec\theta =$	$\pm\frac{1}{\sqrt{1-\sin^2\theta}}$	$\frac{1}{\cos\theta}$	$\pm\sqrt{1+\tan^2\theta}$	$\sec\theta$	$\pm\frac{\sqrt{1+\cot^2\theta}}{\cot\theta}$	$\pm\frac{\csc\theta}{\sqrt{\csc^2-1}}$
$\cot\theta =$	$\pm\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\pm\frac{\cos\theta}{\pm\sqrt{1-\cos^2\theta}}$	$\frac{1}{\tan\theta}$	$\pm\frac{1}{\sqrt{\sec^2\theta-1}}$	$\cot\theta$	$\pm\sqrt{\csc^2\theta-1}$
$\csc\theta =$	$\frac{1}{\sin\theta}$	$\pm\frac{1}{\sqrt{1-\cos^2\theta}}$	$\pm\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$	$\pm\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}$	$\pm\sqrt{1+\cot^2\theta}$	$\csc$

3.5 Calculus

3.5.1 Limits

3.5.2 Properties

- $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$
- $\lim_{x\to a}[f(x)\pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
- $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x)\lim_{x\to a} g(x)$
- $$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \qquad \lim_{x\to a} g(x) \neq 0$$
- $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$

Useful Limits

- $\lim_{x\to\infty} e^x = \infty$
- $\lim_{x\to-\infty} e^x = 0$
- $\lim_{x\to\infty} \ln(x) = \infty$
- $\lim_{x\to 0^-} \ln(x) = -\infty$
- $\lim_{x\to 0} x \log x = 0$

L'Hôpital's rule

- $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Squeeze principle

- For $g(x) \le f(x) \le h(x)$ and $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$:
  $\lim_{x\to a} f(x) = L$

3.5.3 Differentiation

First Principles

- $\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

Nature of derivatives

Derivative	**Function**
$f'x > 0$	Increasing
$f'x = 0$	Stationary
$f'x < 0$	Decreasing

Second Derivative	**Function**	**Stationary Points [$f'x = 0$]**
$f''x > 0$	Concave up	Local Minimum
$f'x = 0$	No information	Inflection Point
$f''x < 0$	Concave down	Local Maximum

Product Rule

- $(uv)' = uv' + vu'$
- $\frac{\mathrm{d}}{\mathrm{d}x} f(x)gx = f(x)g'(x) + f'(x)g(x)$

Quotient Rule

- $(\frac{u}{v})' = \frac{vu' - uv'}{v^2}$
- $\frac{\mathrm{d}}{\mathrm{d}x} \frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$

Chain Rule

- $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \frac{\mathrm{d}u}{\mathrm{d}x}$
- $\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = f'(g(x))g'(x)$

Useful Derivatives

- $\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$
- $\frac{\mathrm{d}}{\mathrm{d}x}a^x = a^x \ln(a)$
- $\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x$
- $\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x},\ x > 0$
- $\frac{\mathrm{d}}{\mathrm{d}x}\ln |x| = \frac{1}{x},\ x \neq 0$
- $\frac{\mathrm{d}}{\mathrm{d}x}\ln(f(x)) = \frac{f'x}{f(x)}$
- $\frac{\mathrm{d}}{\mathrm{d}x}\log_b x = \frac{1}{x\ln(b)},\ x > 0$
- $\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$
- $\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$
- $\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$
- $\frac{\mathrm{d}}{\mathrm{d}x}\sec x = \sec x \tan x$
- $\frac{\mathrm{d}}{\mathrm{d}x}\csc x = -\csc x \cot x$
- $\frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\csc^2 x$
- $\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$
- $\frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1} x = \frac{1}{-\sqrt{1-x^2}}$
- $\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1} x = \frac{1}{-\sqrt{1+x^2}}$

3.5.4 Partial Differentiation

First Principles

- $\frac{\partial}{\partial x}f(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$

Jacobian Matrix

- $D\vec{f}(\vec{a}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{a}) & \frac{\partial f_1}{\partial x_2}(\vec{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\vec{a}) \\ \frac{\partial f_2}{\partial x_1}(\vec{a}) & \frac{\partial f_2}{\partial x_2}(\vec{a}) & \cdots & \frac{\partial f_2}{\partial x_n}(\vec{a}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\vec{a}) & \frac{\partial f_m}{\partial x_2}(\vec{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\vec{a}) \end{bmatrix}$

Definition of differentiability of a multivariable function

- $$\lim_{\vec{x}\to\vec{a}} = \frac{||f(\vec{x}) - f(\vec{a}) - Df(\vec{a})\cdot(\vec{x}-\vec{a})||}{||\vec{x}-\vec{a}||} = 0$$

3.5.5 The Differential

- $$\mathrm{d}F = \frac{\partial F}{\partial x}\,\mathrm{d}x + \frac{\partial F}{\partial y}\,\mathrm{d}y + \frac{\partial F}{\partial z}\,\mathrm{d}z$$

3.5.6 Line Element

- $$\mathrm{d}S^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

3.5.7 Integration

Properties

- $\int f(x) \pm g(x)\,\mathrm{d}x = (\int f(x)\,\mathrm{d}x \int \pm g(x)\,\mathrm{d}x$
- $\int\limits_a^a f(x)\,\mathrm{d}x = 0$
- $\int\limits_a^b f(x)\,\mathrm{d}x = -\int\limits_b^a f(x)\,\mathrm{d}x$
- $|\int\limits_a^b f(x)\,\mathrm{d}x| \leq \int\limits_a^b |f(x)|\,\mathrm{d}x$
- If $f(x) \geq g(x)$ over $[a,b]$, $\int\limits_a^b f(x)\,\mathrm{d}x \geq \int\limits_b^a g(x)\,\mathrm{d}x$
- If $f(x) \geq 0$ over $[a,b]$, $\int\limits_a^b f(x)\,\mathrm{d}x > 0$
- If $m \leq f(x) \leq M$ over $[a,b]$, $m(b-a) \leq \int\limits_a^b f(x)\,\mathrm{d}x \leq M(b-a)$

Integration by Parts

- $\int u'v = uv - \int uv'$
- $\int f'(x)g(x)\,\mathrm{d}x = f(x)g(x) - \int f(x)g'(x)\,\mathrm{d}x$
- $\int\limits_a^b f'(x)g(x)\,\mathrm{d}x = [f(x)g(x)]_a^b - \int\limits_a^b f(x)g'(x)\,\mathrm{d}x$

Integration by Substitution

- $u = g(x);\ \ \mathrm{d}x = g'(x)\,\mathrm{d}x;\ \int\limits_a^b f(g(x))g'(x)\,\mathrm{d}x = \int\limits_{g(a)}^{g(b)} f(x)\,\mathrm{d}x$

Approximations

Trapezoid rule

- $$\int\limits_a^b f(x)\,\mathrm{d}x \approx \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

Simpson's rule

- $$\int\limits_a^b f(x)\,\mathrm{d}x \approx \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

Useful Indefinite Integrals

- $\int k \, \mathrm{d}x = kx + C$
- $\int \log_b x \, \mathrm{d}x = x(\log_b x - \log_b(e)) + C = x(\log_b x - \dfrac{1}{\ln b}) + C$
- $\int \ln x \, \mathrm{d}x = x(\ln x - 1) + C$
- $\int e^x \, \mathrm{d}x = e^x + C$
- $\int x^n \, \mathrm{d}x = \dfrac{x^{n+1}}{n+1} + C, \ n \neq -1$
- $\int \dfrac{1}{x} \, \mathrm{d}x = \ln |x| + C$
- $\int \dfrac{1}{ax+b} \, \mathrm{d}x = \dfrac{1}{a} \ln |ax+b| + C$
- $\int \sin x \, \mathrm{d}x = -\cos x + C$
- $\int \cos x \, \mathrm{d}x = \sin x + C$
- $\int \tan x \, \mathrm{d}x = \ln |\sec x| + C$
- $\int \sec x \, \mathrm{d}x = \ln |\sec x + \tan x| + C$
- $\int \sec^2 x \, \mathrm{d}x = \tan x + C$
- $\int \csc^2 x \, \mathrm{d}x = -\cot x + C$
- $\int \sec x \tan x \, \mathrm{d}x = \sec x + C$
- $\int \csc x \cot x \, \mathrm{d}x = -\csc x + C$
- $\int \dfrac{1}{a^2 + x^2} \, \mathrm{d}x = \dfrac{1}{a} \tan^{-1}(\dfrac{x}{a}) + C$
- $\int \dfrac{1}{\sqrt{a^2 + x^2}} \, \mathrm{d}x = \sin^{-1}(\dfrac{x}{a}) + C$

Useful Definite Integrals

- $\int\limits_0^\infty \dfrac{x^3}{e^x - 1} \, \mathrm{d}x = \dfrac{\pi^4}{15}$
- $\int\limits_0^\infty x^n e^{-ax} \, \mathrm{d}x = \dfrac{n!}{a^{n+1}}$

3.5.8 Differential Equations

Wronskian

- $W(f, g) = f(x)g'(x) - g(x)f'(x)$
- $W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$

1-Dimensional Green's Functions

- $a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x).$
- $y(x) = \int\limits_{-\infty}^{\infty} G(x,z)f(z)\,\mathrm{d}z$
- $y_1(x),\ y_2(x)$ are linearly independent solutions to the equation.

Given initial conditions

- $W(z) = \begin{vmatrix} y_1(z) & y_2(z) \\ y_1'(z) & y_2'(z) \end{vmatrix} = y_1y_2' - y_2y_1'$
- $G(x,z) = \begin{cases} 0; & z > x > 0 \\ \dfrac{y_1(z)y_2(x)}{a_2(z)W(z)} - \dfrac{y_1(x)y_2(z)}{a_2(z)W(z)}; & x > z > 0 \end{cases}$
- $y(x) = \int\limits_0^x [\frac{y_1(z)y_2(x)}{a_2(z)W(z)} - \frac{y_1(x)y_2(z)}{a_2(z)W(z)}]\, f(z)\,\mathrm{d}z$

Given homogeneous boundary conditions

- $W_{ab}(z) = \begin{vmatrix} y_a(z) & y_b(z) \\ y_a'(z) & y_b'(z) \end{vmatrix} = y_ay_b' - y_by_a' = (\alpha - \beta)W(z)$
- $G(x,z) = \begin{cases} \dfrac{y_a(x)y_b(z)}{a_2(z)W_{ab}(z)}; & a < x < z < b \\ \\ \dfrac{y_a(z)y_b(x)}{a_2(z)W_{ab}(z)}; & a < z < x < b \end{cases}$
- $y(x) = \int\limits_a^x \frac{y_a(z)y_b(x)}{a_2(z)W_{ab}(z)} f(z)\,\mathrm{d}z + \int\limits_x^b \frac{y_a(x)y_b(z)}{a_2(z)W_{ab}(z)} f(z)\,\mathrm{d}z$

3.5.9 Vector Calculus

Vector derivative

- $\mathrm{grad}(f) = \vec{\nabla} f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$
- $\int\limits_a^b (\vec{\nabla} f)\cdot\ \mathrm{d}\vec{l} = f(a) - f(b)$

The Laplacian

- $\delta f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

Divergence

In Cartesian Coordinates

- $\mathrm{div}(f) = \vec{\nabla}\cdot\vec{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$

In Cylindrical Coordinates

- $\vec{\nabla}\cdot\vec{f} = \frac{1}{r}\frac{\partial}{\partial r}(rf_r) + \frac{1}{r}\frac{\partial f_\phi}{\partial \phi} + \frac{\partial f_z}{\partial z}$

In Spherical Coordinates

- $\vec{\nabla}\cdot\vec{f} = \frac{1}{r}\frac{\partial}{\partial r}(r^2 f_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(f_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial f_\phi}{\partial\phi}$

Curl

In Cartesian Coordinates

- $$\text{curl}(f) = \vec{\nabla} \times \vec{f} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = (\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z})\hat{x} + (\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial z})\hat{y} + (\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y})\hat{z}$$

In Cylindrical Coordinates

- $$\vec{\nabla} \times \vec{f} = (\frac{1}{r}\frac{\partial f_z}{\partial \phi} - \frac{\partial f_\phi}{\partial z})\hat{r} + (\frac{\partial f_r}{\partial z} - \frac{\partial f_z}{\partial r})\hat{\phi} + \frac{1}{r}(\frac{\partial}{\partial r}(r f_\phi) - \frac{\partial f_r}{\partial \phi})\hat{z}$$

In Spherical Coordinates

- $$\vec{\nabla} \times \vec{f} = \frac{1}{r\sin\theta}(\frac{\partial}{\partial \theta}(f_\phi \sin\theta) - \frac{\partial f_\theta}{\partial \phi})\hat{r} + \frac{1}{r}(\frac{1}{\sin\theta}\frac{\partial f_r}{\partial \phi} - \frac{\partial}{\partial r}(r f_\phi)) + \frac{1}{r}(\frac{\partial}{\partial r}(r f_\theta) - \frac{\partial f_r}{\partial \theta})$$

Vector Second Derivatives

- $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{f}) = 0$
- $\vec{\nabla} \times (\vec{\nabla} \times \vec{f}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{f}) - \vec{\nabla}^2 \vec{f}$

Vector Laplacian

- $\vec{\nabla}^2 \vec{f} = \vec{\nabla}(\vec{\nabla} \cdot \vec{f}) - \vec{\nabla} \times (\vec{\nabla} \times \vec{f})$

Stokes' Theorem

- $\iint\limits_S (\vec{\nabla} \times \vec{f}) \cdot d\vec{a} = \oint\limits_B \vec{f} \cdot \, \mathrm{d}\vec{l}$

Divergence Theorem

- $\iiint\limits_V (\vec{\nabla} \cdot \vec{f}) dV = \oint\limits_S \vec{f} \cdot \, \mathrm{d}\vec{a}$

3.5.10 Dirac Delta Function

- $\delta(x) = \begin{cases} 0; & x \neq 0 \\ \infty; & x = 0 \end{cases}$
- $\delta(x-a) = \begin{cases} 0; & x \neq a \\ \infty; & x = a \end{cases}$
- $\int\limits_{-\infty}^{\infty} \delta(x) \, \mathrm{d}x = 1$
- $f(x)\delta(x) = f(0)\delta(x)$

3.5.11 Approximations

- $f(x + \Delta x) \approx f(x) + \Delta x f'(x)$

Chapter 4

Statistics

4.1 Variance

- $$\text{var(x)} = \langle (x - \langle x \rangle)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$$

4.2 Standard Deviation

- $$\sigma(x) = \sqrt{\text{var(x)}} = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

4.2.1 Population Standard Deviation

- $$\sigma = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (x_i - \mu)^2}$$

4.2.2 Sample Standard Deviation

- $$\sigma_{\text{sample}} = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})^2}$$

4.3 Residual Sum of Squares

- $$\text{RSS} = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

4.4 Mean Squared Error

- $$\text{MSE} = \frac{1}{n} \sum_{i=1}^{n} (y_i - \hat{f}(x_i))^2 = \frac{1}{n}\text{RSS} = \frac{1}{n} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

4.5 Residual Standard Error

- $$\text{RSE} = \sqrt{\frac{1}{n-2}\text{RSS}} = \sqrt{\frac{1}{n-2} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2}$$

4.6 Correlation

- $$\text{Cor}(X, Y) = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n} (x_i - \bar{x})^2} \sqrt{\sum_{i=1}^{n} (y_i - \bar{y})^2}}$$

4.7 Distributions

4.7.1 Gaussian / Normal

PDF

- $$\frac{1}{\sqrt{2\pi\sigma^2}}e^{\left(\frac{(x-\mu)^2}{2\sigma^2}\right)}$$

FWHM

- $$\text{FWHM} = \sigma\sqrt{8\ln(2)} \approx 2.355\sigma$$

Appendix A

Constants & Values

A.1 Physics

A.1.1 Physical Constants

c**:** Speed of light

- $= \dfrac{1}{\sqrt{\varepsilon_0 \mu_0}} \quad = 299\ 792\ 458\ m \cdot s^{-1} \quad \approx 3 \times 10^8\ m \cdot s^{-1}$

G**:** Universal gravitational constant

- $= 6.67408(31) \times 10^{-11}\ m^3 \cdot kg^{-1} \cdot s^{-2} \quad \approx 6.67 \times 10^{-11}\ m^3 \cdot kg^{-1} \cdot s^{-2}$

g**:** Average acceleration due to gravity at sea level on Earth

- $= 9.80665\ m \cdot s^{-2} \quad \approx 9.8\ m \cdot s^{-2}$

h**:** Planck constant

- $= 6.626\ 070\ 040 \times 10^{-34} J \cdot s \quad \approx 6.626 \times 10^{-34} J \cdot s$

$\hbar$**:** Reduced Planck constant

- $= \dfrac{h}{2\pi} = 1.054\ 571\ 726 \times 10^{-34} J \cdot s \quad \approx 1.055 \times 10^{-34} J \cdot s$

k_B**:** Boltzmann constant

- $= 1.3\ 806\ 488 \times 10^{-23}\ J \cdot K^{-1} \quad \approx 1.38 \times 10^{-23}\ J \cdot K^{-1}$

k_e**:** Coulomb's constant

- $= \dfrac{1}{4\pi\varepsilon_0} \quad = 8.987\ 551\ 787 \times 10^9\ N \cdot m \cdot C^{-2} \quad \approx 9 \times 10^9\ N \cdot m \cdot C^{-2}$

N_A**:** Avogadro constant

- $= 6.022\ 140\ 857(74) \times 10^{23}\ mol^{-1} \quad \approx 6.022 \times 10^{23}\ mol^{-1}$

α**:** Fine-structure constant

- $= \dfrac{1}{4\pi\varepsilon_0} \dfrac{e^2}{\hbar c} \quad = \dfrac{1}{137.035\ 999\ 139} \quad = 0.007\ 297\ 352\ 5664 \quad \approx \dfrac{1}{137}$

ε_0**:** Vacuum permittivity

- $= \dfrac{1}{\mu_0 c^2} \quad = 8.854\ 187\ 817 \times 10^{-12}\ F \cdot m^{-1} \quad \approx 8.85 \times 10^{-6}\ F \cdot m^{-1}$

μ_0**:** Vacuum permeability

- $= 4\pi \times 10^{-7}\ N \cdot A^{-2} \quad = \dfrac{1}{\varepsilon_0 c^2} \quad \approx 1.257 \times 10^{-6}\ N \cdot A^{-2}$

σ **:** Stefan-Boltzmann constant

- $= \dfrac{\pi^2 k_B^4}{60 \hbar^3 c^2} \quad = 5.670\ 367 \times 10^{-8} W \cdot m^{-2} \cdot K^{-4}$

A.1.2 Useful Quantities

Density of air (ρ_A): $1.2922\ kg \cdot m^{-3}$

Density of water (ρ_w): $10^3\ kg \cdot m^{-3}$

Mass of an electron (m_e): $9.10\ 938\ 291 \times 10^{-31}\ kg \quad \approx 9 \times 10^{-31}\ kg \quad \approx 0.510\ 998\ 9461\ \frac{MeV}{c^2}$

Mass of a neutron (m_n): $1.674\ 927\ 351 \times 10^{-27}\ kg \quad \approx 1.675 \times 10^{-27}\ kg$

$939.565\ 4133 \frac{MeV}{c^2}$

Mass of a proton (m_p): $1.672\ 621\ 777 \times 10^{-27}\ kg \quad \approx 1.672 \times 10^{-27}\ kg$

$938.272\ 0813 \frac{MeV}{c^2}$

Speed of sound in air: $343.2\ m \cdot s^{-1}$

Specific heat capacity of water: $4.186 \times 10^3\ J \cdot kg^{-1} \cdot K^{-1}$

A.2 Astronomy

A.2.1 Useful Quantities

Surface Temperature of the Sun: ($T_\odot$): $= 5778\ K \qquad = 5505\ °C$

Planetary Properties

Body	Mass	Average Radius	Semi-major axis	Eccentricity	Orbital period
Mercury ☿	$3.3011 \times 10^{23}\ kg$ $0.055\ M_\oplus$ $1.66 \times 10^{-7}\ M_\odot$ $1.74 \times 10^{-4}\ M_♃$	$2.4397 \times 10^{6}\ m$ $0.3829\ R_\oplus$	$5.790905 \times 10^{10}\ m$ $0.387098\ AU$	0.205630	0.240856 yr
Venus ♀	$4.8675 \times 10^{24}\ kg$ $0.815\ M_\oplus$ $2.447 \times 10^{-6}\ M_\odot$ $2.56 \times 10^{-3}\ M_♃$	$6.0518 \times 10^{6}\ m$ $0.9499\ R_\oplus$	$1.08208000 \times 10^{11}\ m$ $0.723332\ AU$	0.006772	0.615198 yr
Earth ⊕	$5.97237 \times 10^{24}\ kg$ $1\ M_\oplus$ $3.003 \times 10^{-6}\ M_\odot$ $2.67 \times 10^{-3}\ M_♃$	$6.3710 \times 10^{6}\ m$ $1\ R_\oplus$	$1.49598023 \times 10^{11}\ m$ $1.000001\ AU$	0.0167086	1.000017 yr
Mars ♂	$6.4171 \times 10^{23}\ kg$ $0.107\ M_\oplus$ $3.226 \times 10^{-7}\ M_\odot$ $3.38 \times 10^{-4}\ M_♃$	$3.3895 \times 10^{6}\ m$ $0.53\ R_\oplus$	$2.27939200 \times 10^{11}\ m$ $1.523679\ AU$	0.0934	1.88082 yr
Jupiter ♃	$1.8982 \times 10^{27}\ kg$ $317.8\ M_\oplus$ $9.55 \times 10^{-4}\ M_\odot$ $1\ M_♃$	$6.9911 \times 10^{7}\ m$ $10.97\ R_\oplus$	7.4052×10^{11} m $5.2044\ AU$	0.0489	11.862 yr
Saturn ♄	$5.6834 \times 10^{26}\ kg$ $95.159\ M_\oplus$ $2.86 \times 10^{-4}\ M_\odot$ $0.299\ M_♃$	$5.8232 \times 10^{7}\ m$ $9.14\ R_\oplus$	$1.43353 \times 10^{12}\ m$ $9.5826\ AU$	0.0565	29.4571 yr
Uranus ⛢	$8.68 \times 10^{25}\ kg$ $14.536\ M_\oplus$ $4.36 \times 10^{-5}\ M_\odot$ $0.046\ M_♃$	$2.5362 \times 10^{7}\ m$ $3.98\ R_\oplus$	$2.87504 \times 10^{12}\ m$ $19.2184\ AU$	0.046381	84.0205 yr
Neptune ♆	$1.0243 \times 10^{26}\ kg$ $17.147\ M_\oplus$ $5.15 \times 10^{-5}\ M_\odot$ $0.054\ M_♃$	$2.4622 \times 10^{7}\ m$ $3.86\ R_\oplus$	$4.5 \times 10^{12}\ m$ $30.11\ AU$	0.009456	164.8 yr

A.3 Mathematics

Euler's number (e): $\sum_{n=0}^{n} \frac{1}{n!} = \frac{1}{1} + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \cdots$

$$= 2.71828182845904523536028747135266249775724709369995\ldots$$

$$\approx 2.718$$

Pi (π): $\frac{C}{d} = \frac{C}{2r}$

$$= 3.14159265358979323846264338327950288419716939937510\ldots$$

$$\approx 3.142$$

Appendix B

Units of Measurement

B.1 SI System

Universally acknowledged as the best system of units.

B.1.1 Base Units

Amount of Substance: mole (mol)

- The amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kg of carbon-12.
- $= 6.022\ 140\ 857 \times 10^{23}$

Electric Current: ampere (A)

- The constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 m apart in vacuum, would produce between these conductors a force equal to $210^{-7} N \cdot m^{-1}$ of length.
- $= C \cdot s^{-1}$

Length: metre (m)

- The distance traveled by light in vacuum in $\dfrac{1}{299\ 792\ 458}\ s$
- $= 3.2808\ ft$

Luminous intensity: candela (cd)

- The luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 5.4×10^{14} Hz and that has a radiant intensity in that direction of $\dfrac{1}{683}$ watt per steradian.

Mass: kilogram (kg)

- $= 2.205 lb$

Thermodynamic Temperature: kelvin (K)

- $\frac{1}{273.16}$ of the thermodynamic temperature of the triple point of water.
- $= T[°C] + 273.15$

Time: second (s)

- The duration of 9 192 631 770 periods of rotation corresponding to the two hyperfine levels of the ground-state of the caesium-133 atom.

B.1.2 Derived Units

Angle: radian (rad)

- A full circle divided by 2π.
- $= m \cdot m^{-1} \qquad = \frac{180}{\pi} \qquad = 206265\, arcsecs \qquad \approx 57.3°$

Electric Charge: coulomb (C)

- $= A \cdot s \qquad = 6.242 \times 10^{18}\, e$

Electrical capacitance: farad (F)

- $= m^{-2} \cdot kg^{-1} \cdot s^4 \cdot A^2$

Electrical conductance: siemens (S)

- $= A \cdot V^{-1} \qquad = kg^{-1} \cdot m^{-2} \cdot s^3 \cdot A^2$

Electrical inductance: henry (H)

- $= Wb \cdot A^{-1} \qquad = kg \cdot m^2 \cdot s^{-2} \cdot A^{-2}$

Electrical potential difference / Voltage: volt (V)

- $= W \cdot A^{-1} \qquad = kg \cdot m^2 \cdot s^{-3} \cdot A^{-1}$

Electrical resistance: ohm (Ω)

- $= V \cdot A^{-1} \qquad = kg \cdot m^2 \cdot s^{-3} \cdot A^{-2}$

Energy: joule (J)

- $= N \cdot m \qquad = kg \cdot m^2 \cdot s^{-2}$

Force: newton (N)

- $= kg \cdot m \cdot s^{-2} \qquad = 0.224809\, lbf$

Frequency: hertz (Hz)

- $= s^{-1}$

Illuminance: lux (lx)

- $= lm \cdot m^2 \qquad = m^{-2} \cdot cd$

Luminous flux: lumen (lm)

- $= cd \cdot sr \qquad = cd$

Magnetic flux: weber (Wb)

- $= V \cdot s \qquad = kg \cdot m^2 \cdot s^{-2} \cdot A^{-1}$

Magnetic flux density: tesla (T)

- $= kg \cdot s^{-2} \cdot A^{-1}$

Power: watt (W)

- $= J \cdot s \qquad = kg \cdot m^2 \cdot s^{-3}$

Pressure: pascal (Pa)

- $= N \cdot m^{-2} \qquad = kg \cdot m^{-1} \cdot s^{-2}$

Radioactivity: becquerel (Ω)

- Decays per second
- $= s^{-1}$

Solid angle: steradian (sr)

- $= m^2 \cdot m^{-2}$

Temperature: degree Celcius ($°C$)

- $T[C] = T[K] - 273.15$

B.2 CGS (centimetres-grams-seconds)

Commonly used in astronomy, to everyone's disappointment.

Acceleration: gal (Gal)

- $= cm \cdot s^{-2} \qquad = 10^-2\, m \cdot s^{-2}$

Energy: erg (erg)

- $= g \cdot cm^2 \cdot s^{-2} \qquad = 10^{-7}\ J$

Force: dyne (dyn)

- $= g \cdot cm \cdot s^{-2} \qquad = 10^{-5}\ N$

Length: centimetre (cm)

- $= 0.01\ m$

Mass: gram (g)

- $= 10^{-3}\ kg$

Power: erg per second (erg/s)

- $= g \cdot cm^2 \cdot s^{-2} \qquad = 10^{-7}\ W$

Pressure: barye (Ba)

- $= g \cdot cm^{-1} \cdot s^{-2} \qquad = 10^{-1}\ Pa$

Time: second (s)

Velocity: centimetre per second (cm/s)

- $= 10^{-2}\ m \cdot s^{-1}$

Viscosity (dynamic): poise (P)

- $= g \cdot cm^{-1}\ s^{-1} \qquad = 10^{-1}\ Pa \cdot s$

Viscosity (kinematic): stokes (St)

- $= g \cdot cm^2\ s^{-1} \qquad = 10^{-4}\ m^2 \cdot s^{-1}$

Wavenumber: kayser (K)

- $= cm^{-1} \qquad\qquad = 100\, m^{-1}$

B.3 Natural Units

Handy when you're dealing with small things.

Charge: elementary charge (e)

- The electric charge of a proton.
- $= 1.602\ 176\ 565 \times 10^{-19}\ C \qquad \approx 1.6 \times 10^{-19}\ C$

Energy: electron volt (eV)

- The work done to move an electron across one volt of potential.
- $= e \cdot V = 1.602\ 176\ 565 \times 10^{-19}\ J \qquad \approx 1.6 \times 10^{-19}\ J$

B.4 Planck Units

Units based around the five universal constants c, G, $\hbar$, $k_e = \frac{1}{4\pi\varepsilon_0}$ and k_B. Only the base units are listed; units for other quantities can be easily derived from these.

Planck length (l_P): $=\sqrt{\frac{\hbar G}{c^3}} \quad = 1.616\ 229 \times 10^{-35} m$

Planck mass (l_P): $=\sqrt{\frac{\hbar c}{G}} \quad = 2.176\ 470 \times 10^{-8} kg$

Planck time (l_P): $= \frac{l_P}{c} \quad = \frac{\hbar}{m_P c^2} \quad = \sqrt{\frac{\hbar G}{c^5}} \quad = 5.391\ 16 \times 10^{-44} s$

Planck charge (q_P): $=\sqrt{4\pi\varepsilon_0 \hbar c} \quad = \frac{e}{\sqrt{\alpha}} \quad = 1.875\ 545\ 956 \times 10^{-18} C$

Planck temperature (T_P): $= \frac{m_P c^2}{k_B} \quad = \sqrt{\frac{\hbar c^5}{G k_B^2}} \quad = 1.416\ 808 \times 10^{32} K$

B.5 Astronomy units

B.5.1 Astronomical system

Distance: astronomical unit (AU)

- Roughly the distance from the Earth to the Sun.
- $= 1.4960 \times 10^{11}\ m \quad = 4.8481 \times 10^{-6}\ pc \quad = 1.5813 \times 10^{-5}\ ly$

Mass: solar mass ($M_\odot$)

- $= 1.98855 \times 10^{30}\ kg \approx 2 \times 10^{30}\ kg = 1048\ M_{♃} = 332\ 950\ M_\odot$

Time: Day

- $= 86\ 400 s$

Complimentary units

Distance: Solar radius ($R_\odot$)

- $= 6.957 \times 10^8\ m \quad = 695\ 700\ km \quad \approx 7 \times 10^8\ m$

Distance: parsec (pc)

- The distance at which the parallax of an object over the course of the Earth's orbit is one arcsec.
- $= 3.0857 \times 10^{16}\ m \quad = 2.0626 \times 10^5\ AU \quad = 3.26156\ ly$

Distance: light year (ly)

- The distance travelled by light in a vacuum in a year.
- $= 9.4607 \times 10^{15}\ m \quad = 6.3241 \times 10^4\ AU \quad = 0.3066\ pc$

Mass: Earth mass ($M_\odot$)

- $= 5.9722 \times 10^{24} kg \quad \approx 6 \times 10^{27} kg$

Mass: Jupiter mass ($M_{♃}$)

- $= 1.898 \times 10^{27} kg \quad \approx 1.9 \times 10^{27} kg$

Specific Flux: Jansky (Jy)

- $= 10^{-26}\ W \cdot m^{-2} \cdot Hz^{-1}$

B.5.2 Equatorial Coordinate System

Right Ascension (α)

Hour (^{h}): $\frac{1}{24}\,circle \quad = 15°$

Minute (^{m}): $\frac{1}{60}^h \quad = \frac{1}{1440}\,circle \quad = 15'$

Second (^{s}): $\frac{1}{60}^m \quad = \frac{1}{3600}^h \quad = \frac{1}{86400}\,circle \quad = 15”$

Declination (δ)

Declination is measured using normal degrees (see *Degrees of Angle*) from the equator.

B.6 United States customary units (aka Imperial Units)

B.6.1 Length

Point (p): $= \frac{127}{360}\,mm$

Pica ($P/$): $= 12\,p \quad = \frac{127}{30}\,mm$

Inch (in or $''$): $= 6\,P/ \quad = 25.4\,mm$

Foot (ft or $'$): $= 12\,in \quad = 0.3048\,m$

Yard (yd): $= 3\,ft \quad = 0.9144\,m$

Mile (Mi): $= 5280\,ft \quad = 1760\,yd \quad = 1.609344\,km$

B.7 Degrees of Angle

Degree (°): $\frac{1}{360}\,circle \quad = \frac{\pi}{180}\,rad \quad \approx 0.0174532925199433\,rad$

Minute of arc ($arcmin$ or $'$): $\frac{1}{60}^\circ \quad = \frac{1}{21600}\,circle \quad = \frac{\pi}{10800}\,rad$

Second of arc($arcsec$ or $''$): $\frac{1}{60}\,arcmin \quad = \frac{1}{3600}^\circ \quad = \frac{1}{206265}\,circle \quad = \frac{\pi}{648000}\,rad$

B.8 Miscellaneous Units

Area: barn

- $= 100 fm^2 \quad = 1 \times 10^{-29} m^2$

B.8.1 Pressure

Bar (bar): $= 10^5\,Pa \quad \approx 0.9869\,atm$

Atmosphere (atm): $= 101325\,Pa \quad = 1.01325\,bar$

Torr ($torr$): $= \frac{1}{760}\,atm \quad = \frac{101325}{760}\,Pa \quad \approx 133.3224\,Pa$

B.9 Prefixes

atto (a) **=** $\times 10^{-18}$

femto (f) **=** $\times 10^{-15}$

pico (p) **=** $\times 10^{-12}$

nano (n) **=** $\times 10^{-9}$

micro (μ) **=** $\times 10^{-6}$

milli (m) **=** $\times 10^{-3}$

centi (c) **=** $\times 10^{-2}$

deca (da) **=** $\times 10^{1}$

hecto (h) **=** $\times 10^{2}$

kilo (k) **=** $\times 10^{3}$

mega (M) **=** $\times 10^{6}$

giga (G) **=** $\times 10^{9}$

tera (T) **=** $\times 10^{12}$

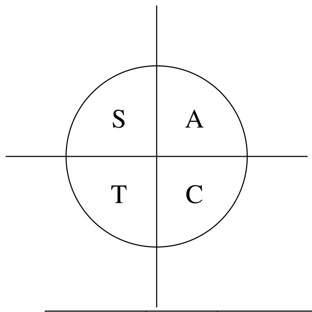
peta (P) **=** $\times 10^{15}$

exa (E) **=** $\times 10^{18}$

Appendix C

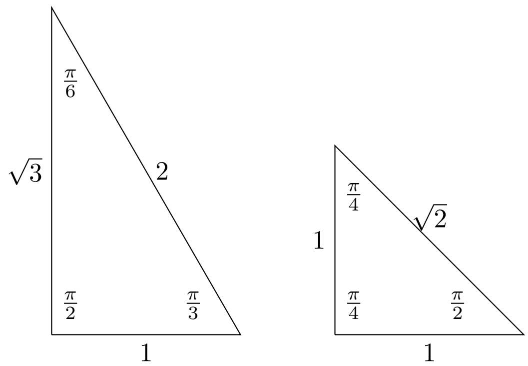
Mathematical Stuff

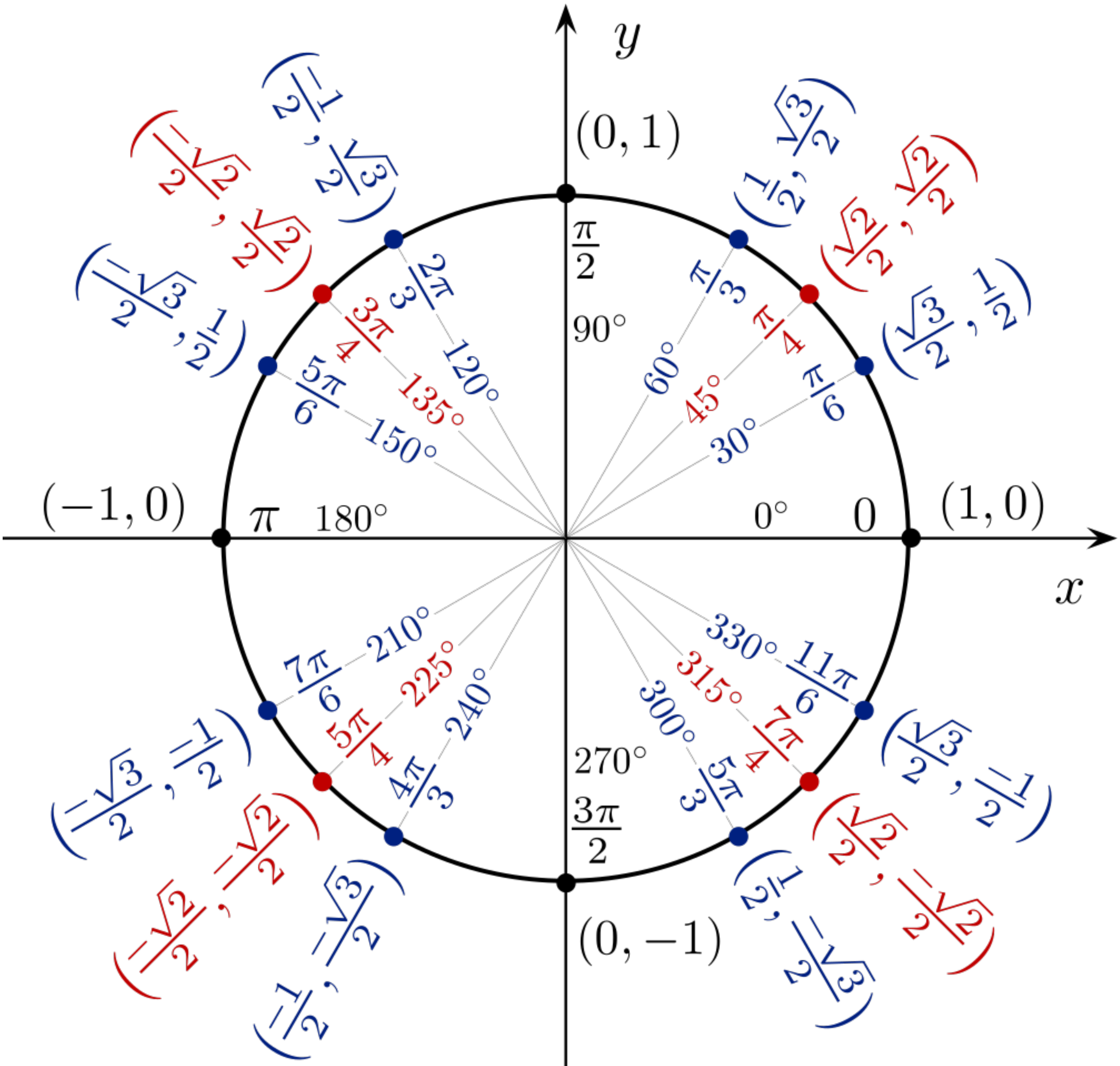
C.1 Trigonometric Values



rad	°	sin	cos	tan
0	0	0	1	0
$\pi/6$	30	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$
$\pi/4$	45	$1/\sqrt{2}$	$1/\sqrt{2}$	1
$\pi/3$	60	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$\pi/2$	90	1	0	$\pm\infty$
$2\pi/3$	120	$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}$
$3\pi/4$	135	$1/\sqrt{2}$	$-1/\sqrt{2}$	-1
$5\pi/6$	150	$1/2$	$-\sqrt{3}/2$	$-1/\sqrt{3}$
π	180	0	-1	0
$7\pi/6$	210	$-1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$
$5\pi/4$	225	$-1/\sqrt{2}$	$-1/\sqrt{2}$	1
$4\pi/3$	240	$-\sqrt{3}/2$	$-1/2$	$\sqrt{3}$
$3\pi/2$	270	-1	0	$\pm\infty$
$5\pi/3$	300	$-\sqrt{3}/2$	$1/2$	$-\sqrt{3}$
$7\pi/4$	315	$-1/\sqrt{2}$	$1/\sqrt{2}$	-1
$11\pi/6$	330	$-1/2$	$\sqrt{3}/2$	$-1/\sqrt{3}$
2π	360	0	1	0

C.1.1 Pythagorean Triples





y
x
(0, 1)
(1, 0)
(−1, 0)
(0, −1)
π/2
90°
0
0°
π
180°
270°
3π/2
π/6
30°
π/4
45°
π/3
60°
2π/3
120°
3π/4
135°
5π/6
150°
7π/6
210°
5π/4
225°
4π/3
240°
5π/3
300°
7π/4
315°
11π/6
330°
(√3/2, 1/2)
(√2/2, √2/2)
(1/2, √3/2)
(−1/2, √3/2)
(−√2/2, √2/2)
(−√3/2, 1/2)
(−√3/2, −1/2)
(−√2/2, −√2/2)
(−1/2, −√3/2)
(1/2, −√3/2)
(√2/2, −√2/2)
(√3/2, −1/2)

Appendix D

Boring stuff

D.1 Licensing

D.2 Contact

Visit www.webofworlds.net for science fiction, science fact, geeky opinions, and maybe some Python code.
Suggestions or corrections are welcome at webofworlds@gmail.com

D.3 Source Code

Available here:
https://github.com/Lachimax/The-Ultimate-Cheat-Sheet-for-Astrophysics-Students

D.4 Version History

v 0.1 2016: This project is begun in a trio of physical exercise books as *The Little Book of Physics Formulae*, *The Little Book of Mathematics Formulae*, and *The Little Book of Astronomy Formulae*

v 0.6 2016: The process of transferring the formulae from paper to LaTeX is initiated, but abandoned (or drifted away from).

v 0.7 2018-03-20: The project is resurrected (probably because the author started MRes), uploaded to Overleaf, and cleaned up.

v 0.8 2018-07-12: Remaining formulae imported from the original books.

v 0.9 2018-07-26: Further formulae imported from undergrad formula sheets.

v 1.0 2018-07-28: First public release, with some additions from 0.9.

v 1.0.1 2018-07-31: Minor corrections, added "dynamical timescale" (2.2.2) and some more formulae to the Statistics chapter (it was looking a little bare).

v 1.1 2019-11-29: Font change; added Planck units and some other miscellaneous units to Units of Measurement; fine structure constant to Physical Constants (why wasn't it already there?); stellar luminosity; formulae for Green's functions and other differential equation techniques; minor corrections; Gaussian distributions to Statistics section; more detailed unit circle.

D.5 Credits

Unit Circle: By Jim.belk [CC BY-SA 3.0 (https://creativecommons.org/licenses/by-sa/3.0) or GFDL (http://www.gnu.org/copyleft/fdl.html)], from Wikimedia Commons (https://commons.wikimedia.org/wiki/File:Unit_circle_angles_color.svg)

Periodic Table: By Dmarcus100 [CC BY-SA 4.0 (https://creativecommons.org/licenses/by-sa/4.0)], from Wikimedia Commons (https://commons.wikimedia.org/wiki/File:Periodic_Table_Of_Elements_Atomic_Mass_Black_And_White.jpg)

1 **H** Hydrogen 1.008																		2 **He** Helium 4.003
3 **Li** Lithium 6.94	4 **Be** Beryllium 9.012												5 **B** Boron 10.81	6 **C** Carbon 12.011	7 **N** Nitrogen 14.007	8 **O** Oxygen 15.999	9 **F** Fluorine 18.998	10 **Ne** Neon 20.180
11 **Na** Sodium 22.990	12 **Mg** Magnesium 24.305												13 **Al** Aluminum 26.982	14 **Si** Silicon 28.085	15 **P** Phosphorus 30.974	16 **S** Sulfur 32.06	17 **Cl** Chlorine 35.45	18 **Ar** Argon 39.948
19 **K** Potassium 39.098	20 **Ca** Calcium 40.078		21 **Sc** Scandium 44.956	22 **Ti** Titanium 47.867	23 **V** Vanadium 50.942	24 **Cr** Chromium 51.996	25 **Mn** Manganese 54.938	26 **Fe** Iron 55.845	27 **Co** Cobalt 58.933	28 **Ni** Nickel 58.693	29 **Cu** Copper 63.546	30 **Zn** Zinc 65.38	31 **Ga** Gallium 69.723	32 **Ge** Germanium 72.630	33 **As** Arsenic 74.922	34 **Se** Selenium 78.97	35 **Br** Bromine 79.904	36 **Kr** Krypton 83.798
37 **Rb** Rubidium 85.468	38 **Sr** Strontium 87.62		39 **Y** Yttrium 88.906	40 **Zr** Zirconium 91.224	41 **Nb** Niobium 92.906	42 **Mo** Molybdenum 95.95	43 **Tc** Technetium [97]	44 **Ru** Ruthenium 101.07	45 **Rh** Rhodium 102.906	46 **Pd** Palladium 106.42	47 **Ag** Silver 107.868	48 **Cd** Cadmium 112.414	49 **In** Indium 114.818	50 **Sn** Tin 118.710	51 **Sb** Antimony 121.760	53 **Te** Tellurium 127.60	53 **I** Iodine 126.904	54 **Xe** Xenon 131.293
55 **Cs** Cesium 132.905	56 **Ba** Barium 137.327	* 57 - 70	71 **Lu** Lutetium 174.967	72 **Hf** Hafnium 178.49	73 **Ta** Tantalum 180.948	74 **W** Tungsten 183.84	75 **Re** Rhenium 186.207	76 **Os** Osmium 190.23	78 **Ir** Iridium 192.217	79 **Pt** Platinum 195.084	80 **Au** Gold 196.997	81 **Hg** Mercury 200.592	81 **Tl** Thallium 204.38	82 **Pb** Lead 207.2	83 **Bi** Bismuth 208.980	84 **Po** Polonium [209]	85 **At** Astatine [210]	86 **Rn** Radon [222]
87 **Fr** Francium [223]	88 **Ra** Radium [226]	** 89 - 102	103 **Lr** Lawrencium [262]	104 **Rf** Rutherfordium [267]	105 **Db** Dubnium [270]	106 **Sg** Seaborgium [269]	107 **Bh** Bohrium [270]	108 **Hs** Hassium [270]	109 **Mt** Meitnerium [278]	110 **Ds** Darmstadtium [281]	111 **Rg** Roentgenium [281]	112 **Cn** Copernicium [285]	113 **Nh** Nihonium [286]	114 **Fl** Flerovium [289]	115 **Mc** Moscovium [289]	116 **Lv** Livermorium [293]	117 **Ts** Tennessine [293]	118 **Og** Oganesson [294]

***Lanthanide series**	57 **La** Lanthanum 138.905	58 **Ce** Cerium 140.116	59 **Pr** Praseodymium 140.908	60 **Nd** Neodymium 144.242	61 **Pm** Promethium [145]	62 **Sm** Samarium 150.36	63 **Eu** Europium 151.964	64 **Gd** Gadolinium 157.25	65 **Tb** Terbium 158.925	66 **Dy** Dysprosium 162.500	67 **Ho** Holmium 164.930	68 **Er** Erbium 167.259	69 **Tm** Thulium 168.934	70 **Yb** Ytterbium 173.045
****Actinide series**	89 **Ac** Actinium [227]	90 **Th** Thorium 232.038	91 **Pa** Protactinium 231.036	92 **U** Uranium 238.029	93 **Np** Neptunium [237]	94 **Pu** Plutonium [244]	95 **Am** Americium [243]	96 **Cm** Curium [247]	97 **Bk** Berkelium [247]	98 **Cf** Californium [251]	99 **Es** Einsteinium [252]	100 **Fm** Fermium [257]	101 **Md** Mendelevium [258]	102 **No** Nobelium [259]