

A Web of Worlds presents
The Ultimate Cheat Sheet for Astrophysics Students

Lachlan Marnoch
www.webofworlds.net

v 1.1.1
June 11, 2022

Contents

1	Physics	5
1.1	Motion	5
1.1.1	Velocity	5
1.1.2	Acceleration	5
1.1.3	Newton's Laws	5
1.1.4	Momentum	5
1.1.5	Centripetal Force	5
1.1.6	Kinetic Energy	5
1.1.7	Projectile Motion	6
1.1.8	Rotation	6
1.1.9	Euler-Lagrange and the Hamiltonian	7
1.2	Oscillations	7
1.2.1	Springs	7
1.3	Materials	8
1.3.1	Density	8
1.4	Energy	8
1.4.1	Work	8
1.5	Forces	8
1.5.1	Buoyancy (Archimedes' Principle)	8
1.5.2	Friction	8
1.6	Waves	8
1.6.1	Wavelength	8
1.6.2	Angular Frequency	8
1.7	Newtonian Gravity	8
1.7.1	Force of Gravity	8
1.7.2	Gravitational Potential (potential energy per unit mass)	8
1.7.3	Gravitational field	8
1.7.4	Gravitational Potential Energy	8
1.7.5	Kepler's Third Law	9
1.8	Electromagnetism	9
1.8.1	Notation	9
1.8.2	Maxwell's Equations	9
1.8.3	Lorentz Force	9
1.8.4	Electric Field	9
1.8.5	Dipole moment	10
1.8.6	Electric potential	10
1.8.7	Electric potential difference	10
1.8.8	Electric potential energy	10
1.8.9	Charge densities	10
1.8.10	Current densities	10
1.8.11	Circuits	10
1.8.12	Capacitors	11
1.8.13	Magnetic fields	11
1.8.14	Inductors	12
1.8.15	Materials	12
1.9	Special Relativity	13
1.9.1	Interval	13
1.9.2	Four-vectors	14
1.9.3	Frames of Reference	14

1.10	General Relativity	15
1.10.1	Metrics	15
1.10.2	Rindler coordinates	15
1.10.3	Einstein summation notation	16
1.10.4	Christoffel symbols	16
1.10.5	Covariant derivatives	16
1.10.6	Riemann curvature tensor	17
1.10.7	Ricci curvature tensor	17
1.10.8	Einstein's equations	17
1.11	Thermodynamics	17
1.11.1	Ideal Gases	17
1.11.2	Microstates	17
1.11.3	Entropy	17
1.11.4	Black bodies	17
1.12	Quantum Mechanics	18
1.12.1	The Uncertainty Principle	18
1.12.2	Bras and Kets	18
1.12.3	Rules for an Inner Product	18
1.12.4	The Born Rule	18
1.12.5	Expectation	18
1.12.6	Variance	18
1.12.7	Standard Deviation	18
1.12.8	Trace	18
1.12.9	Partial Trace	19
1.12.10	The Schrödinger Equation	19
1.12.11	Heisenberg equation of motion	19
1.12.12	Operators	19
2	Astrophysics & Astronomy	21
2.1	Photometry	21
2.1.1	Redshift	21
2.1.2	Apparent magnitude	21
2.1.3	Absolute magnitude	21
2.1.4	Relative magnitudes	21
2.1.5	Flux-magnitude relationship	21
2.1.6	Color	21
2.1.7	Metallicity	21
2.2	Stars	21
2.2.1	Stellar Structure Equations	21
2.2.2	Luminosity	22
2.2.3	Timescales	22
2.2.4	Gravitational potential energy	22
2.2.5	Eddington Limit (hydrostatic equilibrium)	22
2.2.6	Mass-Luminosity Relationship	22
2.3	Galaxies	23
2.3.1	Hubble Elliptical Galaxy Classification	23
2.3.2	Sérsic Profile	23
2.3.3	Density of stars in the Milky Way Galaxy	23
2.4	Cosmology	23
2.4.1	Critical density	23
2.4.2	Density parameter	23
2.5	Black Holes	23
2.5.1	Schwarzschild Radius	23
2.6	Instrumentation	23
2.6.1	Lensmaker's equation	23
2.6.2	Focal ratio / Focal number	23
2.6.3	Field of view	23
2.6.4	Resolution Limits	23
2.6.5	Nyquist sampling	24
2.6.6	Plate scale	24
2.6.7	Fitting error	24

2.6.8	Adaptive optics error	24
2.6.9	Signal-to-noise ratio	24
2.6.10	Atmospheric Extinction	24
2.6.11	Rocket science	24
3	Mathematics	25
3.1	Notation	25
3.2	Algebra	25
3.2.1	Factorisation	25
3.2.2	Absolute Value	25
3.2.3	Quadratics	25
3.2.4	Logarithms	26
3.2.5	Vectors	26
3.2.6	Factorials	26
3.2.7	Inner product definition	27
3.2.8	Complex Numbers	27
3.2.9	Power Series	27
3.2.10	Matrix Operations	27
3.2.11	Matrix Types	28
3.2.12	Change of Basis Unitary	30
3.2.13	Commutator	31
3.2.14	Anticommutator	31
3.2.15	Cauchy-Schwarz Inequality	31
3.3	Geometry	31
3.3.1	Pythagorean theorem	31
3.3.2	Properties of shapes	31
3.3.3	Properties of solids	31
3.3.4	Circular formulae	31
3.3.5	Useful Functions	32
3.3.6	Coordinates	33
3.3.7	Hyperbolic Functions	33
3.4	Trigonometry	34
3.4.1	Identities	34
3.5	Calculus	37
3.5.1	Limits	37
3.5.2	Properties	37
3.5.3	Differentiation	38
3.5.4	Partial Differentiation	39
3.5.5	The Differential	40
3.5.6	Line Element	40
3.5.7	Integration	40
3.5.8	Differential Equations	41
3.5.9	Vector Calculus	42
3.5.10	Dirac Delta Function	43
3.5.11	Approximations	43
4	Statistics	44
4.1	Variance	44
4.2	Standard Deviation	44
4.2.1	Population Standard Deviation	44
4.2.2	Sample Standard Deviation	44
4.3	Residual Sum of Squares	44
4.4	Mean Squared Error	44
4.5	Residual Standard Error	44
4.6	Correlation	44
4.7	Distributions	45
4.7.1	Gaussian / Normal	45

A	Constants & Values	46
A.1	Physics	46
A.1.1	Physical Constants	46
A.1.2	Useful Quantities	47
A.2	Astronomy	48
A.2.1	Useful Quantities	48
A.3	Mathematics	48
B	Units of Measurement	49
B.1	SI System	49
B.1.1	Base Units	49
B.1.2	Derived Units	50
B.2	CGS (centimetres-grams-seconds)	51
B.3	Natural Units	51
B.4	Planck Units	52
B.5	Astronomy units	52
B.5.1	Astronomical system	52
B.5.2	Equatorial Coordinate System	53
B.6	United States customary units (aka Imperial Units)	53
B.6.1	Length	53
B.7	Degrees of Angle	53
B.8	Miscellaneous Units	53
B.8.1	Pressure	53
B.9	Prefixes	54
C	Mathematical Stuff	55
C.1	Trigonometric Values	55
C.1.1	Pythagorean Triples	55
D	Boring stuff	57
D.1	Licensing	57
D.2	Contact	57
D.3	Source Code	57
D.4	Version History	57
D.5	Credits	58

Chapter 1

Physics

1.1 Motion

1.1.1 Velocity

- $$\vec{v} = \frac{\Delta \vec{x}}{\Delta t} = \frac{d\vec{x}}{dt} = \dot{\vec{x}}$$

1.1.2 Acceleration

- $$\vec{a} = \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d^2 \vec{x}}{dt^2} = \ddot{\vec{x}}$$

1.1.3 Newton's Laws

Newton's First Law

- When viewed in an inertial reference frame, an object either remains at rest or continues to move at a constant velocity, unless acted upon by a net force.

Newton's Second Law

- $$\vec{F}_{net} = m\vec{a} = \frac{d\vec{p}}{dt}$$

Newton's Third Law

- $$\vec{F}_A = -\vec{F}_B$$
- When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.

1.1.4 Momentum

- $$\vec{p} = \gamma m \vec{v} \approx m \vec{v}$$
- $$\Delta \vec{p} = \vec{F} \Delta t$$
- $$\vec{F} = \frac{\Delta \vec{p}}{\Delta t} = \frac{d\vec{p}}{dt}$$

1.1.5 Centripetal Force

- $$F_c = \frac{mv^2}{r}$$

1.1.6 Kinetic Energy

- $$K = \frac{1}{2}mv^2$$

1.1.7 Projectile Motion

- $v_y^2 = u_y^2 + 2a_y \Delta y$
- $x = u_x t$
- $\Delta y = u_y \Delta t + \frac{1}{2} a_y \Delta t^2 = u_y t + \frac{1}{2} \frac{F_y}{m} \Delta t^2$

1.1.8 Rotation

Angular Velocity

- $\omega = \frac{d\theta}{dt} = \dot{\theta}$
- $\omega = \frac{v}{r}$
- $\vec{v} = \vec{r} \times \vec{\omega}$

Angular Acceleration

- $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \dot{\omega} = \ddot{\theta}$

Moment of Inertia

Point Mass

- $I = mr^2$

Several Point Masses

- $I = \sum mr^2$

Continuous mass

- $I = \int r^2 dm$

Parallel axis theorem

- $I = I_{com} + md^2$

Thin disc rotating about centre

- $I = \frac{MR^2}{2}$

Thin hoop rotating about centre

- $I = MR^2$

Thin rod rotating about centre

- $I = \frac{ML^2}{12}$

Thin rod rotating about end

- $I = \frac{ML^2}{3}$

Rotational Kinetic Energy

- $K_{rot} = \frac{1}{2} I \omega^2$

Total Kinetic Energy

- $K_{tot} = K_{trans} + K_{rot} = \frac{1}{2} (mr_{com}^2 + I_{com}) \omega^2$

Angular Momentum

- $\vec{L} = I\vec{\omega} = \vec{r} \times \vec{p}$

Torque

- $\vec{\tau} = I\vec{\alpha} = \frac{dL}{dt} = \vec{r} \times \vec{F}$

1.1.9 Euler-Lagrange and the Hamiltonian

Lagrangian

- $\ell = T - V = \sum_{lm} a(q) \dot{q}_l \dot{q}_m$
- $= K(\dot{q}_l) - U(q_l)$

Generalised coordinates & momenta

- $p_k \equiv \frac{\partial L}{\partial \dot{q}_k}$

Euler-Lagrange Equation

- $\frac{d}{dt} \frac{\partial \ell}{\partial \dot{x}} - \frac{\partial \ell}{\partial x} = 0$

Action

- $S[x(t)] = \int_{t_A}^{t_B} \ell(\dot{x}(t), x(t)) dt$

Hamiltonian

- $\mathcal{H} = \sum_l p_l \dot{q} - L$
- $\dot{P} = -\frac{\partial H}{\partial Q}$
- $\dot{Q} = \frac{\partial H}{\partial P}$
- $\dot{P} = -\omega^2 Q$
- $\dot{Q} = P$

1.2 Oscillations

1.2.1 Springs

Force of a Spring

- $\vec{F} = -k_s \vec{x}$

Potential Energy of a Spring

- $U_s = \frac{1}{2} k_s x^2$

Angular Frequency of a Spring

- $\omega = \sqrt{\frac{k_s}{m}}$

1.3 Materials

1.3.1 Density

- $$\rho = \frac{m}{V} = \frac{dm}{dV}$$

1.4 Energy

1.4.1 Work

- $$W = \int_a^b \vec{F} \cdot d\vec{l} \approx \vec{F} \cdot \vec{s}$$

1.5 Forces

1.5.1 Buoyancy (Archimedes' Principle)

- $$F_{buoy} = m_{displaced}g = \rho_d V_d g$$

1.5.2 Friction

- $$F_K \approx \mu_K F_{\perp}$$
- $$F_S \leq \mu_S F_{\perp}$$

1.6 Waves

- $$a \sin(\omega t - kx + \phi)$$
- $$k = \frac{2\pi}{\lambda}$$

1.6.1 Wavelength

- $$v = f\lambda$$

1.6.2 Angular Frequency

- $$\omega = \frac{2\pi}{T} = 2\pi f$$

1.7 Newtonian Gravity

1.7.1 Force of Gravity

- $$\vec{F}_G = \frac{GmM}{r^2} \hat{r} = -m\vec{\nabla}\Phi(\vec{r}) \approx -mg\hat{y} = m\vec{g}$$

1.7.2 Gravitational Potential (potential energy per unit mass)

- $$\Phi(\vec{r}) = -\sum_i \frac{GM(\vec{r}_i)}{|\vec{r} - \vec{r}_i|} = -\int \frac{G\mu(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3r'$$

1.7.3 Gravitational field

- $$\vec{g}(\vec{r}) = \frac{GM}{r^2} = -\nabla\Phi(\vec{r})$$

1.7.4 Gravitational Potential Energy

- $$U_G = -\frac{GmM}{r} \approx mgh$$

1.7.5 Kepler's Third Law

- $\frac{T^2}{r^3} = \frac{4\pi^2}{G(m+M)} = \text{constant}$

1.8 Electromagnetism

1.8.1 Notation

- $\vec{r} = \vec{r} - \vec{r}'$

1.8.2 Maxwell's Equations

	Integral form	Differential form
Gauss's Law	$\oiint_S \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} \iiint_V \rho \, dV$ $= \frac{\sum Q_{\text{enc}}}{\epsilon_0}$	$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$
Gauss's Law for Magnetism	$\oiint_S \vec{B} \cdot d\vec{a} = 0$	$\vec{\nabla} \cdot \vec{B} = 0$
Maxwell-Faraday equation	$\oint_b \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint_S \vec{B} \cdot d\vec{a}$	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
Ampère's circuital law	$\oint_b \vec{B} \cdot d\vec{l} = \mu_0 \iint_S \vec{J} \cdot d\vec{a} + \mu_0 \epsilon_0 \frac{d}{dt} \iint_S \vec{E} \cdot d\vec{a}$ $= \mu_0 (I_{\text{enc}} + \epsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot d\vec{a})$	$\vec{\nabla} \times \vec{B} = \mu_0 (\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t})$

1.8.3 Lorentz Force

On a point charge

- $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

On a current

- $d\vec{F} = I \int d\vec{l} \times \vec{B}$
- $\vec{F} = \vec{I}L \times \vec{B}$

1.8.4 Electric Field

- $\vec{E} = \int_V \frac{\rho(\vec{r}')}{r^2} \hat{z} \, d\tau$

From a single point charge

- $\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$

From a dipole

- $|\vec{E}_{\text{axis}}| \approx \frac{2p}{4\pi\epsilon_0 r^3}$
- $|\vec{E}_{\perp}| \approx \frac{p}{4\pi\epsilon_0 r^3}$

1.8.5 Dipole moment

- $\vec{p} = q\vec{d}$

1.8.6 Electric potential

- $V = \frac{1}{4\pi\epsilon_0} \frac{Q}{z}$
- $\nabla^2 V = \frac{-\rho}{\epsilon_0}$

In a single-point charge field

- $\Delta(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$

1.8.7 Electric potential difference

- $\Delta(\vec{r}) = - \int_{\vec{b}}^{\vec{a}} \vec{E} \cdot d\vec{l}$

In a single-point charge field

- $\Delta(\vec{r}) = \frac{1}{4\pi\epsilon_0} Q \left(\frac{1}{b} - \frac{1}{a} \right)$

1.8.8 Electric potential energy

- $U_E = q\Delta V = \frac{1}{4\pi\epsilon_0} \frac{qQ}{z}$

Energy stored in an electrostatic field distribution

- $U_E = \frac{1}{2} = \epsilon_0 E^2 \times \text{volume}$

1.8.9 Charge densities

Surface

- $\sigma = \frac{dq}{da} = \frac{Q}{A}$

Line

- $\lambda = \frac{dq}{dl} = \frac{Q}{L}$

1.8.10 Current densities

Volume

- $\vec{J} = \frac{d\vec{I}}{d\vec{a}_\perp} = \frac{I}{A_\perp} = \sigma(\vec{E} + \vec{v} \times B) = |q|nu(\vec{E} + \vec{v} \times B)$
- $\vec{\nabla} \cdot \vec{J} = 0$

Surface

- $\vec{K} = \frac{d\vec{I}}{d\vec{l}_\perp} = \frac{I}{l} = \sigma\vec{v}$

1.8.11 Circuits

Electron drift velocity

- $\vec{v} = u\vec{E}_{\text{net}}$

Current per unit charge

- $i = nA_{cs}\bar{v} = nA_{cs}uE_{\text{net}}$

Current

- $I = ei = enA_{cs}uE_{\text{net}} = \frac{dq}{dt}$

Electrical Power

- $P = IV = I^2R$

Voltage (Electric potential difference)

- $V = \Delta V = IR = -\varepsilon$

Electromotive Force (EMF) from a Non-Coulomb force

- $\epsilon = \frac{F_{\text{NCD}}}{e}$

Resistance

- $R = \frac{L\rho}{A} = \frac{L}{\sigma A}$
- $R_{\text{series}} = R_1 + R_2 + \dots + R_n$
- $\frac{1}{R_{\text{parallel}}} = \frac{1}{R_1} + \frac{1}{R_2} + \dots + \frac{1}{R_n}$

1.8.12 Capacitors

Capacitance

- $C = \frac{Q}{V} = \frac{\varepsilon A}{d} = \frac{k\varepsilon_0 A}{d}$

Energy stored in a capacitor

- $W = \frac{CV^2}{2}$

Electric field in a capacitor

- $E = \frac{Q}{\varepsilon_0 A}$

Potential difference across a capacitor

- $\Delta V = -\frac{dQ}{A\varepsilon_0}$

1.8.13 Magnetic fields

- $\vec{B}(\vec{z}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \hat{z}}{z^2} dl$
- $d\vec{B} = \frac{\mu_0}{4\pi} \frac{q\vec{v} \times \hat{z}}{z^2} = \frac{\mu_0}{4\pi} \frac{I d\vec{l} \times \hat{z}}{z^2}$

Magnetic field due to a wire

- $\vec{B} = \frac{\mu_0}{4\pi} \frac{2I}{r} \hat{\phi}$

Magnetic vector potential

- $\vec{A}(\vec{z}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{z} d\tau$
- $\vec{\nabla} \times \vec{A} = \vec{B}$
- $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = -\mu_0 \vec{J}$
- $\vec{\nabla} \cdot \vec{A} = 0$

1.8.14 Inductors

- $\varepsilon = -LI$

Energy stored in an inductor

- $W = \frac{LI^2}{2}$

1.8.15 Materials

Macroscopic Maxwell's Equations (Materials)

	Integral form	Differential form
Gauss's Laws	$\oiint_S \vec{P} \cdot d\vec{a} = -\sum Q_B$	$\vec{\nabla} \cdot \vec{P} = -\rho_B$
	$\oiint_S \vec{D} \cdot d\vec{a} = \sum Q_f$	$\vec{\nabla} \cdot \vec{D} = \rho_f$
Gauss's Law for Magnetism	$\oiint_S \vec{B} \cdot d\vec{a} = 0$	$\vec{\nabla} \cdot \vec{B} = 0$
Maxwell-Faraday equation	$\oint_b \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \oiint_S \vec{B} \cdot d\vec{a}$	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
Ampère's circuital law	$\oint_b \vec{H} \cdot d\vec{l} = I_{f,enc} + \frac{\partial}{\partial t} \oiint_S \vec{D} \cdot d\vec{a}$	$\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$

Dielectric constant

- $k = \frac{\varepsilon}{\varepsilon_0} = \varepsilon_r$
- $\varepsilon = k\varepsilon_0 = \varepsilon_r \varepsilon_0$

Susceptibility

- $\chi_e = 1 - \varepsilon_r$

Polarisability

- $\vec{P} = \varepsilon_0 \chi_e \vec{E} = n\vec{p}$

Bound Charge

Surface

- $\sigma_B = \vec{p} \cdot \hat{n}$

Volume

- $\rho_B = -\vec{\nabla} \cdot \vec{P}$

Total

- $Q_B = \sigma_B + \rho_B = \vec{p} \cdot \hat{n} - \vec{\nabla} \cdot \vec{P}$

Electric displacement

- $\vec{D} = \epsilon \vec{E} = k\epsilon_0 \vec{E} = \epsilon_0 \vec{E} + \vec{P}$

Magnetic field

- $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$

Magnetic dipole

- $\vec{m} = I\vec{a}$

Bound current

- $\vec{J}_B = \vec{\nabla} \times \vec{M}$
- $\vec{K}_B = \vec{M} \times \hat{n}$

1.9 Special Relativity

1.9.1 Interval

- $\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$
- $ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$
- $\Delta s^2 < 0$ is a timelike interval. Events separated by this interval can be causally related.
- $\Delta s^2 = 0$ is a lightlike interval. Events separated by this interval can be causally related, but only by a lightspeed signal.
- $\Delta s^2 > 0$ is a spacelike interval. Events separated by this interval CANNOT be causally related.

Gamma Factor

- $\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$
- $\gamma = \frac{dt}{d\tau}$

Mass-energy

- $E_{\text{rest}} = mc^2$
- $E = \gamma mc^2 = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} mc^2$

Relativistic kinetic energy

- $K = \gamma mc^2 - mc^2$

Length contraction

- $l_v = \frac{l_0}{\gamma} = l_0 \sqrt{1 - \left(\frac{v}{c}\right)^2}$

Time dilation

- $t_v = \gamma t_0 = \frac{t_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$

Mass dilation

- $$m_v = \gamma m_0 = \frac{m_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

Relative Velocity

- $$u'_x = \frac{\Delta x'}{\Delta t} = \frac{u_x - v_x}{1 - \frac{v_x u_x}{c^2}}$$

Relativistic Momentum

- $$\vec{p} = \gamma \vec{v} = \frac{m \vec{v}}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

1.9.2 Four-vectors

Four-space

- $$\mathbf{s} = \mathbf{x} = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

Four-velocity (proper velocity)

- $$\mathbf{u} = \frac{d\mathbf{s}}{d\tau} = \gamma \begin{bmatrix} c \\ v_x \\ v_y \\ v_z \end{bmatrix}$$
- $$\mathbf{u} \cdot \mathbf{u} = -c^2$$

Four-acceleration

- $$\mathbf{w} = \frac{d\mathbf{u}}{d\tau} = \gamma \begin{bmatrix} c \\ v_x \\ v_y \\ v_z \end{bmatrix}$$
- $$\mathbf{w} \cdot \mathbf{u} = 0$$

Four-momentum

- $$\mathbf{p} = \begin{bmatrix} E/c \\ p_x \\ p_y \\ p_z \end{bmatrix} = \gamma m \begin{bmatrix} c \\ v_x \\ v_y \\ v_z \end{bmatrix} = m\mathbf{u}$$

1.9.3 Frames of Reference

Condition for an inertial frame

- $$\frac{d^2x}{dt^2} = \frac{d^2y}{dt^2} = \frac{d^2z}{dt^2} = 0$$

Galilean Transformations

- $$x' = x + vt$$
- $$y' = y$$
- $$z' = z$$
- All assuming x is along the axis of motion and $x = x'$ when $t = 0$.

Lorentz Boosts

- $t' = \gamma \left(t - \frac{vx}{c^2} \right)$
- $x' = \gamma(x - vt)$
- $y' = y$
- $z' = z$
- (x is along the axis of motion)
- $$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

General Lorentz transformation

- $$\begin{bmatrix} b'^0 \\ b'^1 \\ b'^2 \\ b'^3 \end{bmatrix} = \begin{bmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b^0 \\ b^1 \\ b^2 \\ b^3 \end{bmatrix}$$
- Motion along the x -axis.

Proper Time

- $$\tau = \int_{t_A}^{t_B} \frac{1}{\gamma} dt = \int_{t_A}^{t_B} \sqrt{1 - \frac{v^2(t)}{c^2}} dt$$

1.10 General Relativity

1.10.1 Metrics

Minkowski

- $$\eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{bmatrix}$$
- $$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 = -c^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

Schwarzschild

- $$g = \begin{bmatrix} -(1 - \frac{2GM}{c^2 r}) & 0 & 0 & 0 \\ 0 & (1 - \frac{2GM}{c^2 r})^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{bmatrix}$$
- $$ds^2 = -(1 - \frac{2GM}{c^2 r})c^2 dt^2 + (1 - \frac{2GM}{c^2 r})^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

1.10.2 Rindler coordinates

Line element

- $$ds^2 = - \left(1 + \frac{gx'}{c^2} \right)^2 (c dt')^2 + dx'^2$$

1.10.3 Einstein summation notation

- $a_\mu b^\mu \equiv \sum_{\mu=0}^3 a_\mu b^\mu$
- Contravariant: e^α
- Covariant: e_α
- $t_{\alpha\beta} = g_{\beta\gamma} t_{\alpha}{}^\gamma$
- $t_{\alpha}{}^\beta = g^{\beta\gamma} t_{\alpha\gamma}$
- $t'^\alpha{}_\beta = \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{\partial x^\delta}{\partial x'^\beta} t'^\gamma{}_\delta$
- $t'^\alpha{}_\beta = \frac{\partial x^\gamma}{\partial x'^\alpha} \frac{\partial x'^\delta}{\partial x^\beta} t_{\gamma\delta}$

Metrics

- $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$
- $g^{\alpha\beta} = \frac{1}{g_{\alpha\beta}}$
- $\delta_\beta^\alpha = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$
- $\delta_\gamma^\alpha a^\gamma = a^\alpha$
- $g^{\alpha\gamma} g_{\gamma\beta} = \delta_\beta^\alpha$

Four-vector product

- $\mathbf{a} \cdot \mathbf{b} = g_{\alpha\beta} a^\alpha b^\beta = a_\beta b^\beta$

1.10.4 Christoffel symbols

- $\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(\frac{\partial g_{\delta\beta}}{\partial x^\gamma} + \frac{\partial g_{\delta\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\delta} \right)$
- $\Gamma_{\alpha\beta\gamma} = \frac{1}{2} \left(\frac{\partial g_{\delta\beta}}{\partial x^\gamma} + \frac{\partial g_{\delta\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\delta} \right)$
- $\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0$

1.10.5 Covariant derivatives

- $\nabla_\gamma t^\alpha{}_\beta = \frac{\partial t^\alpha{}_\beta}{\partial x^\gamma} + \Gamma^\alpha{}_{\gamma\delta} t^{\delta\beta} - \Gamma^\delta{}_{\gamma\beta} t^{\alpha\delta}$
- $\nabla_\gamma t^{\alpha\beta} = \frac{\partial t^{\alpha\beta}}{\partial x^\gamma} + \Gamma^\alpha{}_{\gamma\delta} t^{\delta\beta} + \Gamma^\beta{}_{\gamma\delta} t^{\alpha\delta}$
- $\nabla_\gamma t_{\alpha\beta} = \frac{\partial t_{\alpha\beta}}{\partial x^\gamma} - \Gamma^\delta{}_{\gamma\alpha} t_{\delta\beta} - \Gamma^\delta{}_{\gamma\beta} t_{\alpha\delta}$
- $\nabla_\gamma t_{\alpha}{}^\beta = \frac{\partial t_{\alpha}{}^\beta}{\partial x^\gamma} - \Gamma^\delta{}_{\gamma\alpha} t_{\delta}{}^\beta + \Gamma^\beta{}_{\gamma\delta} t_{\alpha}{}^\delta$

1.10.6 Riemann curvature tensor

- $R^\alpha{}_{\beta\gamma\delta} = \frac{\partial\Gamma^\alpha{}_{\beta\delta}}{\partial x^\gamma} - \frac{\partial\Gamma^\alpha{}_{\beta\gamma}}{\partial x^\delta} + \Gamma^\alpha{}_{\gamma\epsilon}\Gamma^\epsilon{}_{\beta\delta} - \Gamma^\alpha{}_{\delta\epsilon}\Gamma^\epsilon{}_{\beta\gamma}$
- $R_{\alpha\beta\gamma\delta} = \frac{1}{2} \left(\frac{\partial^2 g_{\alpha\delta}}{\partial x^\beta \partial x^\gamma} - \frac{\partial^2 g_{\alpha\gamma}}{\partial x^\beta \partial x^\delta} - \frac{\partial^2 g_{\beta\delta}}{\partial x^\alpha \partial x^\gamma} \right) + \frac{\partial^2 g_{\beta\gamma}}{\partial x^\alpha \partial x^\delta}$
- $R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}$
- $R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\delta\gamma}$
- $R_{\alpha\beta\gamma\delta} = R_{\delta\gamma\alpha\beta}$
- $R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0$

1.10.7 Ricci curvature tensor

- $R_{\alpha\beta} = R^\gamma{}_{\alpha\gamma\beta}$
- $R = R^\alpha{}_{\alpha}$

1.10.8 Einstein's equations

- $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$

1.11 Thermodynamics

1.11.1 Ideal Gases

Ideal Gas Law

- $pV = Nk_B T$

Heat / Thermal Energy

- $Q = mc\Delta T$

Heat Capacity

- $C = \frac{dQ}{dT}$

Specific Heat Capacity

- $c = \frac{C}{m}$

1.11.2 Microstates

- $\Omega = \frac{(q + N - 1)}{q!(N - 1)}$

1.11.3 Entropy

- $S = k_B \ln \Omega$

1.11.4 Black bodies

Energy of a photon

- $E = hf$

Wien's Displacement Law

- $\lambda_{max} = \frac{b}{T} = (2.8977729 \times 10^{-3}) \frac{1}{T}$

Stefan-Boltzmann Law

- $I = \sigma T^4$

Spectrum

- $B_\lambda(T) = \frac{2hc^2}{\lambda^5} \frac{1}{\exp\left(\frac{hc}{\lambda k_B T} - 1\right)}$
- $B_\nu(T) = \frac{2h\nu}{c^2} \frac{1}{\exp\left(\frac{h\nu}{k_B T} - 1\right)}$

1.12 Quantum Mechanics

1.12.1 The Uncertainty Principle

- $\Delta x \Delta p \geq \frac{\hbar}{2}$
- $\Delta E \Delta t \geq \frac{\hbar}{2}$

1.12.2 Bras and Kets

- $|\psi\rangle = \langle\psi|^\dagger$

1.12.3 Rules for an Inner Product

- $\langle\psi|\phi\rangle \equiv (|\psi\rangle, |\phi\rangle)$
- Symmetric:
 $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$
- Linear in second component
- Anti-linear in first component

1.12.4 The Born Rule

- $P = |\langle\psi|\psi\rangle|^2$

1.12.5 Expectation

- $\langle A \rangle = \int A |\Psi(x, t)|^2 dx$
- $\langle A \rangle = \langle\psi|A|\psi\rangle$

1.12.6 Variance

- $\text{var}(A) = \langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2$

1.12.7 Standard Deviation

- $\delta A = \sqrt{\text{var}(A)} = \sqrt{\langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2}$

1.12.8 Trace

- $\text{Tr}(A) = \sum_j \langle x_j | A | x_j \rangle$

1.12.9 Partial Trace

- $\text{Tr}_B(|a\rangle\langle a| \otimes |b\rangle\langle b|) \equiv |a\rangle\langle a| \text{Tr}(|b\rangle\langle b|)$
- $\text{Tr}(k_{AB}) = \text{Tr}_A(\text{Tr}_B(k_{AB})) = \text{Tr}_B(\text{Tr}_A(k_{AB}))$
- $\rho_B = \text{Tr}_A(\rho_{AB})$
- The partial trace is linear

1.12.10 The Schrödinger Equation

- $i\hbar \frac{\partial}{\partial t} \Psi(r, t) = \hat{H} \Psi(r, t)$
- $-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x, t)}{\partial x^2} + V(x) \Psi(x, t) = i\hbar \frac{\partial \Psi(x, t)}{\partial t}$
- $-\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} + V(x) \psi(x, t) = E \psi(x)$
- $\hat{H} |\Psi(t)\rangle = i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle$

1.12.11 Heisenberg equation of motion

- $\frac{d}{dt} \hat{A}(t) = \frac{i}{\hbar} [\hat{H}, \hat{A}(t)]$

1.12.12 Operators

- $a_{jk} = \langle j|A|k\rangle$

Diagonalizable Operator

- $A = \sum_j \lambda_j |\lambda_j\rangle\langle\lambda_j|$

Normal Operator

- $A = \sum_j |\lambda_j\rangle\langle\lambda_j|$

Eigenstate Operators

- $(|\lambda_k\rangle\langle\lambda_k|)^n = |\lambda_k\rangle\langle\lambda_k|$

Identity

- $I = \sum_j |x_j\rangle\langle x_j|$

Projector

- $P = |\psi\rangle\langle\psi|$

Density operator

- $\rho \equiv \sum_j P_j |\psi_j\rangle\langle\psi_j|$
- Hermitian: $\rho^\dagger = \rho$
- Normalised: $\text{Tr}(\rho) = 1$
- Positive Semi-Definite: $\langle\psi|\rho|\psi\rangle \geq 0, \forall |\psi\rangle \in \mathbf{H}$
- *purity* = $\text{Tr}(\rho^2)$

- $\frac{1}{d} \leq \text{Tr}(\rho^2) \leq 1$
- Pure: $\text{Tr}(\rho^2) = 1$
- Maximally mixed: $\text{Tr}(\rho^2) = \frac{1}{d}$
- $\rho_A = \text{Tr}_B(\rho_{AB})$
- $\langle A \rangle = \text{Tr}(\rho A)$

Pauli Operators

- $\sigma_x = X = |0\rangle\langle 1| + |1\rangle\langle 0| \doteq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- $\sigma_y = Y = i|1\rangle\langle 0| - i|0\rangle\langle 1| \doteq \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
- $\sigma_z = Z = |0\rangle\langle 0| - |1\rangle\langle 1| \doteq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- $I = |0\rangle\langle 0| + |1\rangle\langle 1| \doteq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (Sometimes included)
- $\text{Tr}(X) = \text{Tr}(Y) = \text{Tr}(Z) = 0$
- With respect to Hilbert-Schmidt Inner Product:
 $\|X\| = \|Y\| = \|Z\| = \|I\| = \sqrt{2}$

Properties

- Unitary
- Hermitian
- $\lambda = \pm 1$

Photon Annihilation and Creation Operators

- $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$
- $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$
- $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$
- $\langle\alpha|\hat{a}^\dagger = \alpha^*\langle\alpha|$

Atomic Energy Level Operators (for a two-level approximation)

- $\hat{\sigma}_+ = |e\rangle\langle g|$
- $\hat{\sigma}_- = |g\rangle\langle e|$
- $\hat{\sigma}_z = |e\rangle\langle e| - |g\rangle\langle g|$
- $\hat{\sigma}_+|g\rangle = |e\rangle$
- $\hat{\sigma}_-|e\rangle = |g\rangle$
- $\hat{\sigma}_+|e\rangle = 0$
- $\hat{\sigma}_-|g\rangle = 0$

Chapter 2

Astrophysics & Astronomy

2.1 Photometry

2.1.1 Redshift

- $z = \frac{\lambda_{\text{obs}} - \lambda_{\text{emit}}}{\lambda_{\text{emit}}} = \frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}} - 1 = \frac{f_{\text{emit}} - f_{\text{obs}}}{f_{\text{obs}}} \approx \frac{v}{c} \quad (v \ll c)$
- $1 + z = \frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}} = \frac{f_{\text{emit}}}{f_{\text{obs}}}$

2.1.2 Apparent magnitude

- $m - m_0 = -2.5 \log_{10} \left(\frac{F}{F_0} \right)$

2.1.3 Absolute magnitude

- $M = m - 5 \log_{10} \left(\frac{d}{10} \right)$

2.1.4 Relative magnitudes

- $\frac{I_a}{I_b} = 100^{\frac{(m_b - m_a)}{5}}$

2.1.5 Flux-magnitude relationship

- $F = F_0 \times 10^{-0.4m}$

2.1.6 Color

- $-2.5 \log \left(\frac{F_{f1}}{F_{f2}} \right) = m_{\text{high}} - m_{\text{low}}$

2.1.7 Metallicity

- $Z = \log_{10} \left(\frac{(Fe/H)_*}{(Fe/H)_{\odot}} \right) = \log_{10}(Fe/H)_* - \log_{10}(Fe/H)_{\odot}$

2.2 Stars

2.2.1 Stellar Structure Equations

Hydrostatic Equilibrium

- $\frac{dP}{dr} = \frac{-GM_r \rho(r)}{r^2}$

Mass Conservation

- $\frac{M_r}{r} = 4\pi r^2 \rho$

Energy Equation

- $\frac{dL_r}{dr} = 4\pi r^2 \rho \varepsilon$

Radiative Transport

- $\frac{dT}{dr}|_{rad} = \frac{3}{4ac} \frac{\bar{\kappa} \rho}{T^3} \frac{L_r}{4\pi r^2}$

2.2.2 Luminosity

- $L = 4\pi R^2 \sigma T_{eff}^4$

2.2.3 Timescales

Thermal / Kelvin-Helmholtz Timescale

- $\tau_{KH} = \frac{|U_*|}{L_*} = \frac{GM_*^2}{R_* L_*}$
- $\tau_{KH\odot} \approx 50$ million years

Dynamical Timescale

- $\tau_{dyn} \approx \sqrt{\frac{R^3}{GM}} \approx \sqrt{G\bar{\rho}}$

Nuclear Timescale / Main Sequence Lifespan

- $\tau_N \approx \tau_{\odot} M^{-3} \approx 10^9 \left(\frac{M}{M_{\odot}}\right)^{-3}$

2.2.4 Gravitational potential energy

- $U_* \approx \frac{-GM^2}{R}$

2.2.5 Eddington Limit (hydrostatic equilibrium)

- $L_{edd} = \frac{4\pi GMm_p c}{\sigma T} \approx 3.2 \times 10^4 \left(\frac{M}{M_{\odot}}\right) [L_{\odot}]$
- $M_{edd} = 3.1 * 10^{-5} \left(\frac{L}{L_{\odot}}\right) [M_{\odot}]$

Eddington Rate (mass loss)

- $\dot{M}_{edd} = \frac{L_{edd}}{\eta c^2} \approx 2.4 \times 10^{-8} \left(\frac{M}{M_{\odot}}\right) [M_{\odot}/\text{year}]$

2.2.6 Mass-Luminosity Relationship

- $\frac{L}{L_{\odot}} \approx b \left(\frac{M}{M_{\odot}}\right)^a ; \quad a, b = \begin{cases} 2.3, 0.23 & M < 0.43M_{\odot} \\ 4, 1 & 0.43M_{\odot} < M < 2M_{\odot} \\ 3.5, 1.5 & 2M_{\odot} < M < 20M_{\odot} \\ 1, 32000 & M > 55M_{\odot} \end{cases}$

2.3 Galaxies

2.3.1 Hubble Elliptical Galaxy Classification

- $10 \times \left(\frac{a-b}{a} \right)$

2.3.2 Sérsic Profile

- $I(R) = I_0 \exp\left\{-b\left[\left(\frac{R}{R_e}\right)^{\frac{1}{n}} - 1\right]\right\}$

2.3.3 Density of stars in the Milky Way Galaxy

- $\rho(R, z) = \rho_0 e^{-z/z_0} e^{-R/h}$

2.4 Cosmology

2.4.1 Critical density

- $\rho_c = \frac{3H^2}{8\pi G}$

2.4.2 Density parameter

- $\Omega \equiv \frac{\rho}{\rho_c} = \frac{8\pi G \rho}{3H^2}$

2.5 Black Holes

2.5.1 Schwarzschild Radius

- $r_S = \frac{2GM}{c^2}$

2.6 Instrumentation

2.6.1 Lensmaker's equation

- $\frac{1}{f} = (n-1) \left[\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n-1)d}{nR_1R_2} \right]$
 $\approx (n-1) \left[\frac{1}{R_1} - \frac{1}{R_2} \right]$ (Thin lens approximation)

2.6.2 Focal ratio / Focal number

- $N = \frac{f}{D}$

2.6.3 Field of view

- $FOV = \frac{w_D}{D_T N} = \frac{w_D}{f_{\text{sys}}}$

2.6.4 Resolution Limits

Diffraction limit (Rayleigh criterion)

- $\epsilon_d = 1.22 \frac{\lambda}{D}$

Seeing limit

- $\varepsilon_s = 0.98 \frac{\lambda}{r_0}$

Total Resolution limit

- $\varepsilon \sqrt{\varepsilon_d^2 + \varepsilon_s^2}$

2.6.5 Nyquist sampling

- $\frac{2p}{f_{\text{sys}}} = \frac{\lambda}{D_T}$ (When diffraction limited)
- $N = \frac{2p}{\lambda}$

2.6.6 Plate scale

- $\frac{1}{f} [\text{rad/m}] = \frac{206265}{f} ["/\text{m}]$

2.6.7 Fitting error

- $\sigma_{\text{fit}}^2 = a_f \left(\frac{d_{\text{sub}}}{r_0} \right)^{\frac{5}{3}} = 0.26 \left(\frac{d_{\text{sub}}}{r_0} \right)^{\frac{5}{3}}$

2.6.8 Adaptive optics error

- $\sigma_{\text{total}}^2 = 0.3 \left(\frac{d_{\text{sub}}}{r_0} \right)^{\frac{5}{3}} + \left(\frac{\theta}{\theta_0} \right)^{\frac{5}{3}} + 28.4 \left(\frac{\tau}{\tau_0} \right)^{\frac{5}{3}} + C_{WFS} \left(\frac{\lambda}{F\tau d_{\text{sub}}} \right)^2$

2.6.9 Signal-to-noise ratio

- $\text{SNR} = \frac{Ft}{\sqrt{Ft + (B_s n_p t) + (D n_p t) + (R^2 n_p)}}$

2.6.10 Atmospheric Extinction

- $m_\lambda = m_{\lambda,z} - a_\lambda (\sec z)$

2.6.11 Rocket science

Tsiolkovsky rocket equation

- $\Delta v = v_e \ln \left(\frac{m_0}{m_f} \right)$

Chapter 3

Mathematics

3.1 Notation

- $[f(x)]_b^a = f(a) - f(b)$

3.2 Algebra

3.2.1 Factorisation

- $(a + b)^2 = a^2 + b^2 + 2ab$
- $(a - b)^2 = a^2 + b^2 - 2ab$
- $a^2 - b^2 = (a + b)(a - b)$
- $(a + b)(a + c) = a^2 + (b + c)a + bc$
- $(a + b)^3 = a^3 + 3ab^2 + 3a^2b + b^3$
- $(a - b)^3 = a^3 + 3ab^2 - 3a^2b - b^3$
- $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
- $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
- $a^{2n} - b^{2n} = (a^n - b^n)(a^n + b^n)$

3.2.2 Absolute Value

- $|ab| = |a||b|$
- $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
- $|a + b| \leq |a| + |b|$

3.2.3 Quadratics

Quadratic Formula

For $ax^2 + bx + c = 0$, $a \neq 0$:

- $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- $b^2 - 4ac > 0$ - two real unequal solutions.
- $b^2 - 4ac = 0$ - repeated real solution.
- $b^2 - 4ac < 0$ - two complex solutions.

3.2.4 Logarithms

- $y = \log_b(x); x = b^y$
- $\log_b(xy) = \log_b x + \log_b(y)$
- $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b(y)$
- $\log_b(x^p) = p \log_b x$
- $\log_b(b^x) = x$
- $\log_b(a) = \frac{\log_d(a)}{\log_d(b)}$
- $\log_b(\sqrt[p]{x}) = \frac{1}{p} \log_b x$
- $p \log_b x + q \log_b(y) = \log_b(x^p y^q)$
- $b^{\log_b x} = x$
- $\log_b(b) = 1$
- $\log_b(1) = 0$

3.2.5 Vectors

Dot Product

- $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 + \dots + a_n b_n = ab \cos \theta$

Cross Product

- $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2) \hat{i} + (a_3 b_1 - a_1 b_3) \hat{j} + (a_1 b_2 - a_2 b_1) \hat{k} = ab \sin \theta \hat{n}$
- $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
- $\vec{a} \times \vec{b} = 0$

Vector Product Identities

- $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b})$

3.2.6 Factorials

- $n! = n(n-1)(n-2)\dots(2)(1)$
- $(n+1)! = (n+1)n!$

Stirling's approximation

- $n! \approx n \ln(n) - n + O(\ln(n))$

The Factorial Integral

- $\int_0^{\infty} x^n e^{-x} dx$

3.2.7 Inner product definition

1. Linear in first variable:

$$(\alpha a + \beta b, c) = \alpha(a, c) + \beta(b, c)$$

2. Positive-definite:

$$(a, a) \geq 0, (a, a) = 0 \iff a = 0$$

3. Conjugate symmetrical:

$$(a, b) = (b, a)^*$$

$$(a, b) = (b, a), b, a \in \mathbf{R}$$

3.2.8 Complex Numbers

- $z = a + ib = \Re(z) + \Im(z)i$

Euler's Formula

- $e^{i\theta} = \cos \theta + i \sin \theta$

- $re^{i\theta} = |z|e^{i\theta} = r(\cos \theta + i \sin \theta)$

De Moivre's Formula

- $(\cos x + i \sin x)^n = \cos(nx) + i \sin(nx)$

Complex Modulus

- $r = |z| = |a + ib| = \sqrt{a^2 + b^2} = \sqrt{\Re^2(z) + \Im^2(z)}$

Complex Conjugate

- $\bar{z} = (a + ib)^{\bar{}} = a - ib$

- $(a + ib)(a - ib) = |a + ib|^2$

Complex Argument

- $\theta = \arg(z) = \arctan\left(\frac{a}{b}\right) = \arctan\left(\frac{\Im(z)}{\Re(z)}\right)$

3.2.9 Power Series

- $f(x) = \sum_{n=0}^{\infty} f_n x^n$

Notable Series

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \dots$

- $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

- $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

3.2.10 Matrix Operations

Determinant

- $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Transpose

- $a_{jk}^T = a_{kj}$
- $(AB)^T = B^T A^T$
- Linear: $(A + B)^T = A^T + B^T$
- $(rA)^T = rA^T$

Hermitian Adjoint

- $A^\dagger \equiv (A^*)^T = (A^T)^*$
- $(AB)^\dagger = B^\dagger A^\dagger$
- $(A + B)^\dagger = A^\dagger + B^\dagger$

Trace

- $\text{Tr}(A) = \sum_j^n a_{jj}$
- Cyclic: $\text{Tr}(AB) = \text{Tr}(BA)$
- Linear: $\text{Tr}(A + B) = \text{Tr}(A) + \text{Tr}(B)$
 $\text{Tr}(aB) = a\text{Tr}(B), a \in \mathbf{C}$
- $\text{Tr}(SAS^{-1}) = \text{Tr}(A)$

Hilbert-Schmidt Inner Product

- $(A, B) \equiv \text{Tr}(A^\dagger B)$

Rank-Nullity Theorem

- $\text{rk}(A) = \dim(\ker(A)) + \dim(\text{im}(A))$

3.2.11 Matrix Types

Real Matrix

- $a_{jk} \in \mathbf{R}$

Square Matrix

- $m = n$

Symmetric Matrix

- $A = A^T$
- $a_{jk} = a_{kj}$
- Square

Normal Matrix

- $A^\dagger A = AA^\dagger$
- Square
- Diagonalisable

Diagonal Matrix

- $a_{jk} = 0, j \neq k$
- $\lambda_j = a_{jj}$
- Square
- $e^D = \begin{bmatrix} e^{d_{11}} & 0 & \dots & 0 \\ 0 & e^{d_{11}} & \dots & 0 \end{bmatrix}$

Diagonalisable Matrix

- $A = PDP^{-1}$
- Square

Identity Matrix

- $IA = A$
- $i_{jj} = 1$
- $i_{jk} = 0, j \neq k$
- Real
- Square
- Diagonal
- Symmetric
- Hermitian

Hermitian Matrix

- $H = H^\dagger$
- $h_{jk} = h_{kj}^*$
- $h_{jj} \in \mathbf{R}$
- $\lambda \in \mathbf{R}$
- Square
- Normal
- All real, square matrices are Hermitian

Anti-Hermitian Matrix

- $H = -H^\dagger$
- $H_{jk} = -H_{kj}^*$
- Square

Orthogonal Matrix

- $A^T = A^{-1}$
- $AA^T = I$
- $(AA^T)_{jk} = \delta_{jk}$

Positive Semidefinite

- $A \geq 0$
- $\hat{A}^\dagger = \hat{A}, \hat{A} \geq 0$
- $B = \hat{A}^\dagger \hat{A}$ is positive semidefinite for any linear operator \hat{A}
- Positive semidefinite matrices are Hermitian

Projector

- $\hat{P}^2 = \hat{P}$
- $\lambda = 1$ or 0
- $P_1 P_2 \mapsto \mathbf{H}_1 \cap \mathbf{H}_2$
- Projectors are Hermitian

Real Matrix

- $A = A^*$
- $A_{jk} = A_{jk}^*$

Imaginary Matrix

- $A = -A^*$
- $A_{jk} = -A_{jk}^*$

Symmetric Matrix

- $A = A^T$
- $A_{jk} = A_{kj}$
- Square

Antisymmetric Matrix

- $A = -A^T$
- $a_{jk} = -a_{kj}$
- $a_{jj} = 0$
- Square

Unitary Matrix

- $U^\dagger U = U U^\dagger = I$
- $U^\dagger = U^{-1}$
- $(U^\dagger U)_{jk} = \delta_{jk}$
- Square
- Normal
- Hermitian

3.2.12 Change of Basis Unitary

- $(V)_b = [U^\dagger]_a (V)_a$
- $[U]_a = [(b_0)_a (b_1)_a \dots (b_n)_a]$

3.2.13 Commutator

- $[A, B] = AB - BA$
- $[A, A] = 0$
- $[A + B, C] = [A, C] + [B, C]$
- $[A, BC] = [A, B]C + B[A, C]$

3.2.14 Anticommutator

- $\{A, B\} = AB + BA$

3.2.15 Cauchy-Schwarz Inequality

- $|\langle \vec{u}, \vec{v} \rangle|^2 \leq \langle \vec{u}, \vec{u} \rangle \cdot \langle \vec{v}, \vec{v} \rangle$

3.3 Geometry

3.3.1 Pythagorean theorem

- $a^2 + b^2 = c^2$
- $a = \sqrt{b^2 + c^2}$

Higher dimensions

- $r = \sqrt{x^2 + y^2 + z^2}$
- $r = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} = \sqrt{\vec{r} \cdot \vec{r}}$

Distance between two points

- In two dimensions:

$$- \quad d_{12} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- In higher dimensions:

$$- \quad d_{ab} = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + \dots + (b_n - a_n)^2}$$

3.3.2 Properties of shapes

	Area	Circumference
Circle	πR^2	$2\pi R$
Square	L^2	$4L$

3.3.3 Properties of solids

	Surface Area	Volume
Sphere	$4\pi R^2$	$\frac{4}{3}\pi R^3$

3.3.4 Circular formulae

Arc length

- $l = R\theta$

Area of a sector

- $A = \frac{R^2\theta}{2}$

Area of a segment

- $A = \frac{R^2}{2}(\theta - \sin \theta)$

3.3.5 Useful Functions

Parabola

- $f(x) = a(x - h)^2 + k$
- Vertex at (h, k)
- Up-concave if $a > 0$; down-concave if $a < 0$

- $f(x) = ax^2 + bx + c$
- Vertex at $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$
- Up-concave if $a > 0$; down-concave if $a < 0$

Hyperbola

- $\left(\frac{x - h}{a}\right)^2 - \left(\frac{y - k}{b}\right)^2 = 1$
- Centre at (h, k)
- Asymptotes through centre, slope $\pm \frac{b}{a}$

Circle

- $(x - h)^2 + (y - k)^2 = R^2$
- Centre at (h, k)

Ellipse

- $1 = \left(\frac{x - h}{a}\right)^2 + \left(\frac{y - k}{b}\right)^2$
- Centre at (h, k)
- Vertices a units right/left from the centre and vertices b units up/down from the center.

Sphere

- $R^2 = (x - h)^2 + (y - k)^2 + (z - l)^2$
- Centre at (h, k, l) :

Ball

- $R^2 < (x - h)^2 + (y - k)^2 + (z - l)^2$
- Centre at (h, k, l) :

3.3.6 Coordinates

Transformations to Cartesian coordinates

Cartesian	$x = x$	$y = y$	$z = z$	$dV = dx dy dz$
Polar (2D)	$x = r \cos \phi$	$y = r \sin \phi$	N/A	$dA = r dr d\phi$
Cylindrical	$x = r \cos \phi$	$y = r \sin \phi$	$z = z$	$dV = r dr d\theta dz$
Spherical	$x = r \sin \theta \cos \phi$	$y = r \sin \theta \sin \phi$	$z = r \cos \theta$	$dV = r^2 \sin \theta dr d\theta d\phi$

Transformations from Cartesian coordinates

Cartesian	$x = x$	$y = y$	$z = z$
Polar (2D)	$r = \sqrt{x^2 + y^2}$	$\phi' = \arctan \left \frac{y}{x} \right $ (ϕ depends on quadrant)	N/A
Cylindrical	$r = \sqrt{x^2 + y^2}$	$\phi' = \arctan \left \frac{y}{x} \right $ (ϕ depends on quadrant)	$z = z$
Spherical	$r = \sqrt{x^2 + y^2 + z^2}$	$\phi' = \arctan \left \frac{y}{x} \right $ (ϕ depends on quadrant)	$\theta = \arccos \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)$

3.3.7 Hyperbolic Functions

Hyperbolic Sine

- $\sinh x = \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2e^x} = \frac{1 - e^{-2x}}{2e^{-x}}$
- $\sinh x = -i \sin(ix)$

Hyperbolic Cosine

- $\cosh x = \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2e^x} = \frac{1 + e^{-2x}}{2e^{-x}}$
- $\cosh x = \cos(ix)$

Hyperbolic Tangent

- $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{1 - e^{-2x}}{1 + e^{-2x}}$
- $\tanh x = -i \tan(ix)$

Hyperbolic Cotangent

- $\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x} + 1}{e^{2x} - 1} = \frac{1 + e^{-2x}}{1 - e^{-2x}}$
- $\coth x = i \cot(ix)$

Hyperbolic Secant

- $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} = \frac{2e^x}{e^{2x} + 1} = \frac{2e^{-x}}{1 + e^{-2x}}$
- $\operatorname{sech} x = \sec(ix)$

Hyperbolic Cosecant

- $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} = \frac{2e^x}{e^{2x} - 1} = \frac{2e^{-x}}{1 - e^{-2x}}$
- $\operatorname{csch} x = i \operatorname{csc}(ix)$

Identities

- $\cosh^2 x - \sinh^2 x = 1$
- $\sin \theta \cos \theta = \frac{1}{2} \sin(2\theta)$

3.4 Trigonometry

Definitions

- SOH CAH TOA
- $\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$
- $\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$
- $\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin \theta}{\cos \theta}$
- $\cot \theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{\cos \theta}{\sin \theta}$
- $\sec \theta = \frac{\text{hypotenuse}}{\text{adjacent}}$
- $\csc \theta = \frac{\text{hypotenuse}}{\text{opposite}}$

3.4.1 Identities

Pythagorean Identities

- $\cos^2 \theta + \sin^2 \theta = 1$
- $\tan^2 \theta + 1 = \sec^2 \theta$
- $1 + \cot^2 \theta = \csc^2 \theta$

Reciprocals

- $\sin \theta = \frac{1}{\csc \theta}$
- $\cos \theta = \frac{1}{\sec \theta}$
- $\tan \theta = \frac{1}{\cot \theta}$
- $\cot \theta = \frac{1}{\tan \theta}$
- $\sec \theta = \frac{1}{\cos \theta}$
- $\csc \theta = \frac{1}{\sin \theta}$

As complex exponentials

- $\sin \theta = \frac{e^{ix} - e^{-ix}}{2i}$
- $\cos \theta = \frac{e^{ix} + e^{-ix}}{2}$
- $\tan \theta = \frac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})}$
- $\cot \theta = \frac{i(e^{ix} + e^{-ix})}{e^{ix} - e^{-ix}}$
- $\sec \theta = \frac{2}{e^{ix} + e^{-ix}}$
- $\csc \theta = \frac{2i}{e^{ix} - e^{-ix}}$

Symmetries

- $\sin(-\theta) = -\sin \theta$
- $\cos(-\theta) = \cos \theta$
- $\tan(-\theta) = -\tan \theta$
- $\csc(-\theta) = -\csc \theta$
- $\sec(-\theta) = \sec \theta$
- $\cot(-\theta) = -\cot \theta$

- $\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$
- $\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$
- $\tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta$
- $\csc\left(\frac{\pi}{2} - \theta\right) = \sec \theta$
- $\sec\left(\frac{\pi}{2} - \theta\right) = \csc \theta$
- $\cot\left(\frac{\pi}{2} - \theta\right) = \tan \theta$

- $\sin(\pi - \theta) = \sin \theta$
- $\cos(\pi - \theta) = -\cos \theta$
- $\tan(\pi - \theta) = -\tan \theta$
- $\csc(\pi - \theta) = \csc \theta$
- $\sec(\pi - \theta) = -\sec \theta$
- $\cot(\pi - \theta) = -\cot \theta$

Angle sum and difference formulae

- $\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$
- $\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$
- $\tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta}$

Half-angle formulae

- $\sin^2\left(\frac{\theta}{2}\right) = \frac{1 - \cos \theta}{2}$
- $\cos^2\left(\frac{\theta}{2}\right) = \frac{1 + \cos \theta}{2}$
- $\tan^2\left(\frac{\theta}{2}\right) = \frac{1 - \cos \theta}{1 + \cos \theta}$
- $\tan\left(\frac{\theta}{2}\right) = \frac{\tan \theta}{1 + \sec \theta} = \frac{\sin \theta}{1 + \cos \theta} = \frac{1 - \cos \theta}{\sin \theta} = \csc \theta - \cot \theta$

Double-angle formulae

- $\cos(2\theta) = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta = \cos^2 \theta - \sin^2 \theta$
- $\sin(2\theta) = 2 \sin \theta \cos \theta$
- $\tan(2\theta) = \frac{2 \tan \theta}{1 - \tan^2 \theta}$

Sum to Product

- $\sin \alpha + \sin \beta = 2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)$
- $\sin \alpha - \sin \beta = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right)$
- $\cos \alpha + \cos \beta = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)$
- $\cos \alpha - \cos \beta = -2 \sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right)$

Product to Sum

- $\sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$
- $\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)]$
- $\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \cos(\alpha - \beta)]$
- $\cos \alpha \sin \beta = \frac{1}{2} [\sin(\alpha + \beta) - \sin(\alpha - \beta)]$

Law of Sines

- $\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$

Law of Cosines

- $a^2 = b^2 + c^2 - 2bc \cos \alpha$
- $b^2 = a^2 + c^2 - 2ac \cos \beta$
- $c^2 = a^2 + b^2 - 2ab \cos \gamma$

Law of Tangents

- $\frac{a - b}{a + b} = \frac{\tan\left(\frac{1}{2}[\alpha - \beta]\right)}{\tan\left(\frac{1}{2}[\alpha + \beta]\right)}$
- $\frac{b - c}{b + c} = \frac{\tan\left(\frac{1}{2}[\beta - \gamma]\right)}{\tan\left(\frac{1}{2}[\beta + \gamma]\right)}$
- $\frac{a - c}{a + c} = \frac{\tan\left(\frac{1}{2}[\alpha - \gamma]\right)}{\tan\left(\frac{1}{2}[\alpha + \gamma]\right)}$

Mollweide's Formula

- $$\frac{a+b}{c} = \frac{\cos(\frac{1}{2}[\alpha - \beta])}{\sin(\frac{1}{2}\gamma)}$$

Small-angle approximations

- $\sin \theta \approx \theta$
- $\cos \theta \approx 1 - \frac{\theta^2}{2}$
- $\tan \theta \approx \theta$

Other identities

- $\sin \theta \cos \theta = \frac{1}{2} \sin(2\theta)$
- $\cos^2 \theta = \frac{1}{2} (\cos(2\theta) + 1)$

Averages

- $\overline{\sin x} = \overline{\cos x} = 0$
- $\overline{\sin^2 x} = \overline{\cos^2 x} = \frac{1}{2}$

Table of Identities

In terms of...	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\sec \theta$	$\cot \theta$	$\csc \theta$
$\sin \theta =$	$\sin \theta$	$\pm \sqrt{1 - \cos^2 \theta}$	$\pm \frac{\tan \theta}{\sqrt{1 + \tan^2 \theta}}$	$\pm \frac{\sqrt{\sec^2 \theta - 1}}{\sec \theta}$	$\pm \frac{1}{\sqrt{1 + \cot^2 \theta}}$	$\frac{1}{\csc \theta}$
$\cos \theta =$	$\pm \sqrt{1 - \sin^2 \theta}$	$\cos \theta$	$\pm \frac{1}{\sqrt{1 + \tan^2 \theta}}$	$\frac{1}{\sec \theta}$	$\pm \frac{\cot \theta}{\sqrt{1 + \cot^2 \theta}}$	$\pm \frac{\sqrt{\csc^2 \theta - 1}}{\csc \theta}$
$\tan \theta =$	$\pm \frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}}$	$\pm \frac{\sqrt{1 - \cos^2 \theta}}{\cos \theta}$	$\tan \theta$	$\pm \sqrt{\sec^2 \theta - 1}$	$\frac{1}{\cot \theta}$	$\pm \frac{1}{\sqrt{\csc^2 \theta - 1}}$
$\sec \theta =$	$\pm \frac{1}{\sqrt{1 - \sin^2 \theta}}$	$\frac{1}{\cos \theta}$	$\pm \sqrt{1 + \tan^2 \theta}$	$\sec \theta$	$\pm \frac{\sqrt{1 + \cot^2 \theta}}{\cot \theta}$	$\pm \frac{\csc \theta}{\sqrt{\csc^2 \theta - 1}}$
$\cot \theta =$	$\pm \frac{\sqrt{1 - \sin^2 \theta}}{\sin \theta}$	$\pm \frac{\cos \theta}{\pm \sqrt{1 - \cos^2 \theta}}$	$\frac{1}{\tan \theta}$	$\pm \frac{1}{\sqrt{\sec^2 \theta - 1}}$	$\cot \theta$	$\pm \sqrt{\csc^2 \theta - 1}$
$\csc \theta =$	$\frac{1}{\sin \theta}$	$\pm \frac{1}{\sqrt{1 - \cos^2 \theta}}$	$\pm \frac{\sqrt{1 + \tan^2 \theta}}{\tan \theta}$	$\pm \frac{\sec \theta}{\sqrt{\sec^2 \theta - 1}}$	$\pm \sqrt{1 + \cot^2 \theta}$	$\csc \theta$

3.5 Calculus

3.5.1 Limits

3.5.2 Properties

- $\lim_{x \rightarrow a} cf(x) = c \lim_{x \rightarrow a} f(x)$
- $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}, \quad \lim_{x \rightarrow a} g(x) \neq 0$
- $\lim_{x \rightarrow a} [f(x)]^n = [\lim_{x \rightarrow a} f(x)]^n$

Useful Limits

- $\lim_{x \rightarrow \infty} e^x = \infty$
- $\lim_{x \rightarrow -\infty} e^x = 0$
- $\lim_{x \rightarrow \infty} \ln(x) = \infty$
- $\lim_{x \rightarrow 0^-} \ln(x) = -\infty$
- $\lim_{x \rightarrow 0} x \log x = 0$

L'Hôpital's rule

- $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$

Squeeze principle

- For $g(x) \leq f(x) \leq h(x)$ and $\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L$:

$$\lim_{x \rightarrow a} f(x) = L$$

3.5.3 Differentiation

First Principles

- $\frac{d}{dx} f(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

Nature of derivatives

Derivative	Function
$f'x > 0$	Increasing
$f'x = 0$	Stationary
$f'x < 0$	Decreasing

Second Derivative	Function	Stationary Points [$f'x = 0$]
$f''x > 0$	Concave up	Local Minimum
$f''x = 0$	No information	Inflection Point
$f''x < 0$	Concave down	Local Maximum

Product Rule

- $(uv)' = uv' + vu'$
- $\frac{d}{dx} f(x)g(x) = f(x)g'(x) + f'(x)g(x)$

Quotient Rule

- $\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$
- $\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$

Chain Rule

- $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$
- $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

Useful Derivatives

- $\frac{d}{dx}x^n = nx^{n-1}$
- $\frac{d}{dx}a^x = a^x \ln(a)$
- $\frac{d}{dx}e^x = e^x$
- $\frac{d}{dx} \ln x = \frac{1}{x}, x > 0$
- $\frac{d}{dx} \ln |x| = \frac{1}{x}, x \neq 0$
- $\frac{d}{dx} \ln(f(x)) = \frac{f'(x)}{f(x)}$
- $\frac{d}{dx} \log_b x = \frac{1}{x \ln(b)}, x > 0$
- $\frac{d}{dx} \sin x = \cos x$
- $\frac{d}{dx} \cos x = -\sin x$
- $\frac{d}{dx} \tan x = \sec^2 x$
- $\frac{d}{dx} \sec x = \sec x \tan x$
- $\frac{d}{dx} \csc x = -\csc x \cot x$
- $\frac{d}{dx} \cot x = -\csc^2 x$
- $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx} \cos^{-1} x = \frac{1}{-\sqrt{1-x^2}}$
- $\frac{d}{dx} \tan^{-1} x = \frac{1}{-\sqrt{1+x^2}}$

3.5.4 Partial Differentiation

First Principles

- $\frac{\partial}{\partial x} f(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$

Jacobian Matrix

- $D\vec{f}(\vec{a}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{a}) & \frac{\partial f_1}{\partial x_2}(\vec{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\vec{a}) \\ \frac{\partial f_2}{\partial x_1}(\vec{a}) & \frac{\partial f_2}{\partial x_2}(\vec{a}) & \cdots & \frac{\partial f_2}{\partial x_n}(\vec{a}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\vec{a}) & \frac{\partial f_m}{\partial x_2}(\vec{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\vec{a}) \end{bmatrix}$

Definition of differentiability of a multivariable function

- $$\lim_{\vec{x} \rightarrow \vec{a}} \frac{\|f(\vec{x}) - f(\vec{a}) - Df(\vec{a}) \cdot (\vec{x} - \vec{a})\|}{\|\vec{x} - \vec{a}\|} = 0$$

3.5.5 The Differential

- $$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz$$

3.5.6 Line Element

- $$dS^2 = dx^2 + dy^2 + dz^2$$

3.5.7 Integration

Properties

- $$\int f(x) \pm g(x) dx = \left(\int f(x) dx \right) \pm \left(\int g(x) dx \right)$$
- $$\int_a^a f(x) dx = 0$$
- $$\int_a^b f(x) dx = - \int_b^a f(x) dx$$
- $$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$$
- If $f(x) \geq g(x)$ over $[a, b]$, $\int_a^b f(x) dx \geq \int_a^b g(x) dx$
- If $f(x) \geq 0$ over $[a, b]$, $\int_a^b f(x) dx > 0$
- If $m \leq f(x) \leq M$ over $[a, b]$, $m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$

Integration by Parts

- $$\int u'v = uv - \int uv'$$
- $$\int f'(x)g(x) dx = f(x)g(x) - \int f(x)g'(x) dx$$
- $$\int_a^b f'(x)g(x) dx = [f(x)g(x)]_a^b - \int_a^b f(x)g'(x) dx$$

Integration by Substitution

- $$u = g(x); \quad dx = g'(x) dx; \quad \int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(x) dx$$

Approximations

Trapezoid rule

- $$\int_a^b f(x) dx \approx \frac{\Delta x}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

Simpson's rule

- $$\int_a^b f(x) dx \approx \frac{\Delta x}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

Useful Indefinite Integrals

- $\int k \, dx = kx + C$
- $\int \log_b x \, dx = x(\log_b x - \log_b(e)) + C = x(\log_b x - \frac{1}{\ln b}) + C$
- $\int \ln x \, dx = x(\ln x - 1) + C$
- $\int e^x \, dx = e^x + C$
- $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$
- $\int \frac{1}{x} \, dx = \ln|x| + C$
- $\int \frac{1}{ax+b} \, dx = \frac{1}{a} \ln|ax+b| + C$
- $\int \sin x \, dx = -\cos x + C$
- $\int \cos x \, dx = \sin x + C$
- $\int \tan x \, dx = \ln|\sec x| + C$
- $\int \sec x \, dx = \ln|\sec x + \tan x| + C$
- $\int \sec^2 x \, dx = \tan x + C$
- $\int \csc^2 x \, dx = -\cot x + C$
- $\int \sec x \tan x \, dx = \sec x + C$
- $\int \csc x \cot x \, dx = -\csc x + C$
- $\int \frac{1}{a^2+x^2} \, dx = \frac{1}{a} \tan^{-1}(\frac{x}{a}) + C$
- $\int \frac{1}{\sqrt{a^2+x^2}} \, dx = \sin^{-1}(\frac{x}{a}) + C$

Useful Definite Integrals

- $\int_0^{\infty} \frac{x^3}{e^x - 1} \, dx = \frac{\pi^4}{15}$
- $\int_0^{\infty} x^n e^{-ax} \, dx = \frac{n!}{a^{n+1}}$

3.5.8 Differential Equations

Wronskian

- $W(f, g) = f(x)g'(x) - g(x)f'(x)$
- $W(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$

1-Dimensional Green's Functions

- $a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x)$.
- $y(x) = \int_{-\infty}^{\infty} G(x, z)f(z) dz$
- $y_1(x), y_2(x)$ are linearly independent solutions to the equation.

Given initial conditions

- $W(z) = \begin{vmatrix} y_1(z) & y_2(z) \\ y_1'(z) & y_2'(z) \end{vmatrix} = y_1y_2' - y_2y_1'$
- $G(x, z) = \begin{cases} 0; & z > x > 0 \\ \frac{y_1(z)y_2(x)}{a_2(z)W(z)} - \frac{y_1(x)y_2(z)}{a_2(z)W(z)}; & x > z > 0 \end{cases}$
- $y(x) = \int_0^x \left[\frac{y_1(z)y_2(x)}{a_2(z)W(z)} - \frac{y_1(x)y_2(z)}{a_2(z)W(z)} \right] f(z) dz$

Given homogeneous boundary conditions

- $W_{ab}(z) = \begin{vmatrix} y_a(z) & y_b(z) \\ y_a'(z) & y_b'(z) \end{vmatrix} = y_a y_b' - y_b y_a' = (\alpha - \beta)W(z)$
- $G(x, z) = \begin{cases} \frac{y_a(x)y_b(z)}{a_2(z)W_{ab}(z)}; & a < x < z < b \\ \frac{y_a(z)y_b(x)}{a_2(z)W_{ab}(z)}; & a < z < x < b \end{cases}$
- $y(x) = \int_a^x \frac{y_a(z)y_b(x)}{a_2(z)W_{ab}(z)} f(z) dz + \int_x^b \frac{y_a(x)y_b(z)}{a_2(z)W_{ab}(z)} f(z) dz$

3.5.9 Vector Calculus

Vector derivative

- $\text{grad}(f) = \vec{\nabla} f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} + \frac{\partial f}{\partial z} \hat{z}$
- $\int_a^b (\vec{\nabla} f) \cdot d\vec{l} = f(b) - f(a)$

The Laplacian

- $\delta f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

Divergence

In Cartesian Coordinates

- $\text{div}(f) = \vec{\nabla} \cdot \vec{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$

In Cylindrical Coordinates

- $\vec{\nabla} \cdot \vec{f} = \frac{1}{r} \frac{\partial}{\partial r} (r f_r) + \frac{1}{r} \frac{\partial f_\phi}{\partial \phi} + \frac{\partial f_z}{\partial z}$

In Spherical Coordinates

- $\vec{\nabla} \cdot \vec{f} = \frac{1}{r} \frac{\partial}{\partial r} (r^2 f_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (f_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial f_\phi}{\partial \phi}$

Curl

In Cartesian Coordinates

- $$\text{curl}(f) = \vec{\nabla} \times \vec{f} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z} \right) \hat{x} + \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x} \right) \hat{y} + \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right) \hat{z}$$

In Cylindrical Coordinates

- $$\vec{\nabla} \times \vec{f} = \left(\frac{1}{r} \frac{\partial f_z}{\partial \phi} - \frac{\partial f_\phi}{\partial z} \right) \hat{r} + \left(\frac{\partial f_r}{\partial z} - \frac{\partial f_z}{\partial r} \right) \hat{\phi} + \frac{1}{r} \left(\frac{\partial}{\partial r} (r f_\phi) - \frac{\partial f_r}{\partial \phi} \right) \hat{z}$$

In Spherical Coordinates

- $$\vec{\nabla} \times \vec{f} = \frac{1}{r \sin \theta} \left(\frac{\partial}{\partial \theta} (f_\phi \sin \theta) - \frac{\partial f_\theta}{\partial \phi} \right) \hat{r} + \frac{1}{r} \left(\frac{1}{\sin \theta} \frac{\partial f_r}{\partial \phi} - \frac{\partial}{\partial r} (r f_\phi) \right) \hat{\theta} + \frac{1}{r} \left(\frac{\partial}{\partial r} (r f_\theta) - \frac{\partial f_r}{\partial \theta} \right) \hat{\phi}$$

Vector Second Derivatives

- $$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{f}) = 0$$
- $$\vec{\nabla} \times (\vec{\nabla} \times \vec{f}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{f}) - \vec{\nabla}^2 \vec{f}$$

Vector Laplacian

- $$\vec{\nabla}^2 \vec{f} = \vec{\nabla}(\vec{\nabla} \cdot \vec{f}) - \vec{\nabla} \times (\vec{\nabla} \times \vec{f})$$

Stokes' Theorem

- $$\iint_S (\vec{\nabla} \times \vec{f}) \cdot d\vec{a} = \oint_B \vec{f} \cdot d\vec{l}$$

Divergence Theorem

- $$\iiint_V (\vec{\nabla} \cdot \vec{f}) dV = \oint_S \vec{f} \cdot d\vec{a}$$

3.5.10 Dirac Delta Function

- $$\delta(x) = \begin{cases} 0; & x \neq 0 \\ \infty; & x = 0 \end{cases}$$
- $$\delta(x - a) = \begin{cases} 0; & x \neq a \\ \infty; & x = a \end{cases}$$
- $$\int_{-\infty}^{\infty} \delta(x) dx = 1$$
- $$f(x)\delta(x) = f(0)\delta(x)$$

3.5.11 Approximations

- $$f(x + \Delta x) \approx f(x) + \Delta x f'(x)$$

Chapter 4

Statistics

4.1 Variance

- $\text{var}(x) = \langle (x - \langle x \rangle)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$

4.2 Standard Deviation

- $\sigma(x) = \sqrt{\text{var}(x)} = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$

4.2.1 Population Standard Deviation

- $\sigma = \sqrt{\frac{1}{N} \sum_{i=1}^N (x_i - \mu)^2}$

4.2.2 Sample Standard Deviation

- $\sigma_{\text{sample}} = \sqrt{\frac{1}{N-1} \sum_{i=1}^N (x_i - \bar{x})^2}$

4.3 Residual Sum of Squares

- $\text{RSS} = \sum_{i=1}^n (y_i - \hat{y}_i)^2$

4.4 Mean Squared Error

- $\text{MSE} = \frac{1}{n} \sum_{i=1}^n (y_i - \hat{f}(x_i))^2 = \frac{1}{n} \text{RSS} = \frac{1}{n} \sum_{i=1}^n (y_i - \hat{y}_i)^2$

4.5 Residual Standard Error

- $\text{RSE} = \sqrt{\frac{1}{n-2} \text{RSS}} = \sqrt{\frac{1}{n-2} \sum_{i=1}^n (y_i - \hat{y}_i)^2}$

4.6 Correlation

- $\text{Cor}(X, Y) = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2} \sqrt{\sum_{i=1}^n (y_i - \bar{y})^2}}$

4.7 Distributions

4.7.1 Gaussian / Normal

PDF

- $$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\left(\frac{(x-\mu)^2}{2\sigma^2}\right)}$$

FWHM

- $$\text{FWHM} = \sigma\sqrt{8\ln(2)} \approx 2.355\sigma$$

Appendix A

Constants & Values

A.1 Physics

A.1.1 Physical Constants

c : Speed of light

$$\bullet = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 299\,792\,458 \text{ ms}^{-1} \approx 3 \times 10^8 \text{ ms}^{-1}$$

G : Universal gravitational constant

$$\bullet = 6.67408(31) \times 10^{-11} \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2} \approx 6.67 \times 10^{-11} \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}$$

g : Average acceleration due to gravity at sea level on Earth

$$\bullet = 9.80665 \text{ m} \cdot \text{s}^{-2} \approx 9.8 \text{ m} \cdot \text{s}^{-2}$$

h : Planck constant

$$\bullet = 6.626\,070\,040 \times 10^{-34} \text{ J} \cdot \text{s} \approx 6.626 \times 10^{-34} \text{ J} \cdot \text{s}$$

\hbar : Reduced Planck constant

$$\bullet = \frac{h}{2\pi} = 1.054\,571\,726 \times 10^{-34} \text{ J} \cdot \text{s} \approx 1.055 \times 10^{-34} \text{ J} \cdot \text{s}$$

k_B : Boltzmann constant

$$\bullet = 1.3\,806\,488 \times 10^{-23} \text{ J} \cdot \text{K}^{-1} \approx 1.38 \times 10^{-23} \text{ J} \cdot \text{K}^{-1}$$

k_e : Coulomb's constant

$$\bullet = \frac{1}{4\pi\epsilon_0} = 8.987\,551\,787 \times 10^9 \text{ N} \cdot \text{m} \cdot \text{C}^{-2} \approx 9 \times 10^9 \text{ N} \cdot \text{m} \cdot \text{C}^{-2}$$

N_A : Avogadro constant

$$\bullet = 6.022\,140\,857(74) \times 10^{23} \text{ mol}^{-1} \approx 6.022 \times 10^{23} \text{ mol}^{-1}$$

α : Fine-structure constant

$$\bullet = \frac{1}{4\pi\epsilon_0} \frac{e^2}{\hbar c} = \frac{1}{137.035\,999\,139} = 0.007\,297\,352\,5664 \approx \frac{1}{137}$$

ϵ_0 : Vacuum permittivity

$$\bullet = \frac{1}{\mu_0 c^2} = 8.854\,187\,817 \times 10^{-12} \text{ F} \cdot \text{m}^{-1} \approx 8.85 \times 10^{-12} \text{ F} \cdot \text{m}^{-1}$$

μ_0 : Vacuum permeability

$$\bullet = 4\pi \times 10^{-7} \text{ N} \cdot \text{A}^{-2} = \frac{1}{\epsilon_0 c^2} \approx 1.257 \times 10^{-6} \text{ N} \cdot \text{A}^{-2}$$

σ : Stefan-Boltzmann constant

$$\bullet = \frac{\pi^2 k_B^4}{60 \hbar^3 c^2} = 5.670\,367 \times 10^{-8} \text{ W} \cdot \text{m}^{-2} \cdot \text{K}^{-4}$$

A.1.2 Useful Quantities

Density of air (ρ_A): $1.2922 \text{ kg} \cdot \text{m}^{-3}$

Density of water (ρ_w): $10^3 \text{ kg} \cdot \text{m}^{-3}$

Mass of an electron (m_e): $9.10938291 \times 10^{-31} \text{ kg} \approx 9 \times 10^{-31} \text{ kg} \approx 0.5109989461 \frac{\text{MeV}}{c^2}$

Mass of a neutron (m_n): $1.674927351 \times 10^{-27} \text{ kg} \approx 1.675 \times 10^{-27} \text{ kg}$
 $939.5654133 \frac{\text{MeV}}{c^2}$

Mass of a proton (m_p): $1.672621777 \times 10^{-27} \text{ kg} \approx 1.672 \times 10^{-27} \text{ kg}$
 $938.2720813 \frac{\text{MeV}}{c^2}$

Speed of sound in air: $343.2 \text{ m} \cdot \text{s}^{-1}$

Specific heat capacity of water: $4.186 \times 10^3 \text{ J} \cdot \text{kg}^{-1} \cdot \text{K}^{-1}$

A.2 Astronomy

A.2.1 Useful Quantities

Surface Temperature of the Sun: (T_{\odot}): = 5778 K = 5505 °C

Planetary Properties

Body	Mass	Average Radius	Semi-major axis	Eccentricity	Orbital period
Mercury ☿	$3.3011 \times 10^{23} \text{ kg}$ $0.055 M_{\oplus}$ $1.66 \times 10^{-7} M_{\odot}$ $1.74 \times 10^{-4} M_{\text{Jup}}$	$2.4397 \times 10^6 \text{ m}$ $0.3829 R_{\oplus}$	$5.790905 \times 10^{10} \text{ m}$ 0.387098 AU	0.205630	0.240856 yr
Venus ♀	$4.8675 \times 10^{24} \text{ kg}$ $0.815 M_{\oplus}$ $2.447 \times 10^{-6} M_{\odot}$ $2.56 \times 10^{-3} M_{\text{Jup}}$	$6.0518 \times 10^6 \text{ m}$ $0.9499 R_{\oplus}$	$1.08208000 \times 10^{11} \text{ m}$ 0.723332 AU	0.006772	0.615198 yr
Earth ⊕	$5.97237 \times 10^{24} \text{ kg}$ $1 M_{\oplus}$ $3.003 \times 10^{-6} M_{\odot}$ $2.67 \times 10^{-3} M_{\text{Jup}}$	$6.3710 \times 10^6 \text{ m}$ $1 R_{\oplus}$	$1.49598023 \times 10^{11} \text{ m}$ 1.000001 AU	0.0167086	1.000017 yr
Mars ♂	$6.4171 \times 10^{23} \text{ kg}$ $0.107 M_{\oplus}$ $3.226 \times 10^{-7} M_{\odot}$ $3.38 \times 10^{-4} M_{\text{Jup}}$	$3.3895 \times 10^6 \text{ m}$ $0.53 R_{\oplus}$	$2.27939200 \times 10^{11} \text{ m}$ 1.523679 AU	0.0934	1.88082 yr
Jupiter ♃	$1.8982 \times 10^{27} \text{ kg}$ $317.8 M_{\oplus}$ $9.55 \times 10^{-4} M_{\odot}$ $1 M_{\text{Jup}}$	$6.9911 \times 10^7 \text{ m}$ $10.97 R_{\oplus}$	$7.4052 \times 10^{11} \text{ m}$ 5.2044 AU	0.0489	11.862 yr
Saturn ♄	$5.6834 \times 10^{26} \text{ kg}$ $95.159 M_{\oplus}$ $2.86 \times 10^{-4} M_{\odot}$ $0.299 M_{\text{Jup}}$	$5.8232 \times 10^7 \text{ m}$ $9.14 R_{\oplus}$	$1.43353 \times 10^{12} \text{ m}$ 9.5826 AU	0.0565	29.4571 yr
Uranus ♅	$8.68 \times 10^{25} \text{ kg}$ $14.536 M_{\oplus}$ $4.36 \times 10^{-5} M_{\odot}$ $0.046 M_{\text{Jup}}$	$2.5362 \times 10^7 \text{ m}$ $3.98 R_{\oplus}$	$2.87504 \times 10^{12} \text{ m}$ 19.2184 AU	0.046381	84.0205 yr
Neptune ♆	$1.0243 \times 10^{26} \text{ kg}$ $17.147 M_{\oplus}$ $5.15 \times 10^{-5} M_{\odot}$ $0.054 M_{\text{Jup}}$	$2.4622 \times 10^7 \text{ m}$ $3.86 R_{\oplus}$	$4.5 \times 10^{12} \text{ m}$ 30.11 AU	0.009456	164.8 yr

A.3 Mathematics

Euler's number (e): $\sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{1} + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \dots$
 = 2.71828182845904523536028747135266249775724709369995...
 ≈ 2.718

Pi (π): $\frac{C}{d} = \frac{C}{2r}$
 = 3.14159265358979323846264338327950288419716939937510...
 ≈ 3.142

Appendix B

Units of Measurement

B.1 SI System

Universally acknowledged as the best system of units.

B.1.1 Base Units

Amount of Substance: mole (*mol*)

- The amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 *kg* of carbon-12.
- $= 6.022\ 140\ 857 \times 10^{23}$

Electric Current: ampere (*A*)

- The constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 m apart in vacuum, would produce between these conductors a force equal to $2 \times 10^{-7} N \cdot m^{-1}$ of length.
- $= C \cdot s^{-1}$

Length: metre (*m*)

- The distance traveled by light in vacuum in $\frac{1}{299\ 792\ 458} s$
- $= 3.2808 ft$

Luminous intensity: candela (*cd*)

- The luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 5.4×10^{14} Hz and that has a radiant intensity in that direction of $\frac{1}{683}$ watt per steradian.

Mass: kilogram (*kg*)

- $= 2.205 lb$

Thermodynamic Temperature: kelvin (*K*)

- $\frac{1}{273.16}$ of the thermodynamic temperature of the triple point of water.
- $= T[^\circ C] + 273.15$

Time: second (*s*)

- The duration of 9 192 631 770 periods of rotation corresponding to the two hyperfine levels of the ground-state of the caesium-133 atom.

B.1.2 Derived Units

Angle: radian (*rad*)

- A full circle divided by 2π .
- $= m \cdot m^{-1} = \frac{180}{\pi} = 206265 \text{ arcsecs} \approx 57.3^\circ$

Electric Charge: coulomb (*C*)

- $= A \cdot s = 6.242 \times 10^{18} e$

Electrical capacitance: farad (*F*)

- $= m^{-2} \cdot kg^{-1} \cdot s^4 \cdot A^2$

Electrical conductance: siemens (*S*)

- $= A \cdot V^{-1} = kg^{-1} \cdot m^{-2} \cdot s^3 \cdot A^2$

Electrical inductance: henry (*H*)

- $= Wb \cdot A^{-1} = kg \cdot m^2 \cdot s^{-2} \cdot A^{-2}$

Electrical potential difference / Voltage: volt (*V*)

- $= W \cdot A^{-1} = kg \cdot m^2 \cdot s^{-3} \cdot A^{-1}$

Electrical resistance: ohm (Ω)

- $= V \cdot A^{-1} = kg \cdot m^2 \cdot s^{-3} \cdot A^{-2}$

Energy: joule (*J*)

- $= N \cdot m = kg \cdot m^2 \cdot s^{-2}$

Force: newton (*N*)

- $= kg \cdot m \cdot s^{-2} = 0.224809 \text{ lbf}$

Frequency: hertz (*Hz*)

- $= s^{-1}$

Illuminance: lux (*lx*)

- $= lm \cdot m^{-2} = m^{-2} \cdot cd$

Luminous flux: lumen (*lm*)

- $= cd \cdot sr = cd$

Magnetic flux: weber (*Wb*)

- $= V \cdot s = kg \cdot m^2 \cdot s^{-2} \cdot A^{-1}$

Magnetic flux density: tesla (*T*)

- $= kg \cdot s^{-2} \cdot A^{-1}$

Power: watt (*W*)

- $= J \cdot s = kg \cdot m^2 \cdot s^{-3}$

Pressure: pascal (*Pa*)

- $= N \cdot m^{-2} = kg \cdot m^{-1} \cdot s^{-2}$

Radioactivity: becquerel (Ω)

- Decays per second
- $= s^{-1}$

Solid angle: steradian (*sr*)

- $= m^2 \cdot m^{-2}$

Temperature: degree Celcius ($^\circ C$)

- $T[C] = T[K] - 273.15$

B.2 CGS (centimetres-grams-seconds)

Commonly used in astronomy, to everyone's disappointment.

Acceleration: gal (*Gal*)

- $= cm \cdot s^{-2} = 10^{-2} m \cdot s^{-2}$

Energy: erg (*erg*)

- $= g \cdot cm^2 \cdot s^{-2} = 10^{-7} J$

Force: dyne (*dyn*)

- $= g \cdot cm \cdot s^{-2} = 10^{-5} N$

Length: centimetre (*cm*)

- $= 0.01 m$

Mass: gram (*g*)

- $= 10^{-3} kg$

Power: erg per second (*erg/s*)

- $= g \cdot cm^2 \cdot s^{-2} = 10^{-7} W$

Pressure: barye (*Ba*)

- $= g \cdot cm^{-1} \cdot s^{-2} = 10^{-1} Pa$

Time: second (*s*)

Velocity: centimetre per second (*cm/s*)

- $= 10^{-2} m \cdot s^{-1}$

Viscosity (dynamic): poise (*P*)

- $= g \cdot cm^{-1} s^{-1} = 10^{-1} Pa \cdot s$

Viscosity (kinematic): stokes (*St*)

- $= g \cdot cm^2 s^{-1} = 10^{-4} m^2 \cdot s^{-1}$

Wavenumber: kayser (*K*)

- $= cm^{-1} = 100 m^{-1}$

B.3 Natural Units

Handy when you're dealing with small things.

Charge: elementary charge (*e*)

- The electric charge of a proton.
- $= 1.602\,176\,565 \times 10^{-19} C \approx 1.6 \times 10^{-19} C$

Energy: electron volt (*eV*)

- The work done to move an electron across one volt of potential.
- $= e \cdot V = 1.602\,176\,565 \times 10^{-19} J \approx 1.6 \times 10^{-19} J$

B.4 Planck Units

Units based around the five universal constants c , G , \hbar , $k_e = \frac{1}{4\pi\epsilon_0}$ and k_B . Only the base units are listed; units for other quantities can be easily derived from these.

$$\text{Planck length } (l_P): = \sqrt{\frac{\hbar G}{c^3}} = 1.616\,229 \times 10^{-35} m$$

$$\text{Planck mass } (l_P): = \sqrt{\frac{\hbar c}{G}} = 2.176\,470 \times 10^{-8} kg$$

$$\text{Planck time } (l_P): = \frac{l_P}{c} = \frac{\hbar}{m_P c^2} = \sqrt{\frac{\hbar G}{c^5}} = 5.391\,16 \times 10^{-44} s$$

$$\text{Planck charge } (q_P): = \sqrt{4\pi\epsilon_0 \hbar c} = \frac{e}{\sqrt{\alpha}} = 1.875\,545\,956 \times 10^{-18} C$$

$$\text{Planck temperature } (T_P): = \frac{m_P c^2}{k_B} = \sqrt{\frac{\hbar c^5}{G k_B^2}} = 1.416\,808 \times 10^{32} K$$

B.5 Astronomy units

B.5.1 Astronomical system

Distance: astronomical unit (AU)

- Roughly the distance from the Earth to the Sun.
- $= 1.4960 \times 10^{11} m = 4.8481 \times 10^{-6} pc = 1.5813 \times 10^{-5} ly$

Mass: solar mass (M_\odot)

- $= 1.98855 \times 10^{30} kg \approx 2 \times 10^{30} kg = 1048 M_\oplus = 332\,950 M_\odot$

Time: Day

- $= 86\,400 s$

Complimentary units

Distance: Solar radius (R_\odot)

- $= 6.957 \times 10^8 m = 695\,700 km \approx 7 \times 10^8 m$

Distance: parsec (pc)

- The distance at which the parallax of an object over the course of the Earth's orbit is one arcsec.
- $= 3.0857 \times 10^{16} m = 2.0626 \times 10^5 AU = 3.26156 ly$

Distance: light year (ly)

- The distance travelled by light in a vacuum in a year.
- $= 9.4607 \times 10^{15} m = 6.3241 \times 10^4 AU = 0.3066 pc$

Mass: Earth mass (M_\oplus)

- $= 5.9722 \times 10^{24} kg \approx 6 \times 10^{27} kg$

Mass: Jupiter mass (M_\Jup)

- $= 1.898 \times 10^{27} kg \approx 1.9 \times 10^{27} kg$

Specific Flux: Jansky (Jy)

- $= 10^{-26} W \cdot m^{-2} \cdot Hz^{-1}$

B.5.2 Equatorial Coordinate System

Right Ascension (α)

$$\text{Hour } (^h): \frac{1}{24} \text{ circle} = 15^\circ$$

$$\text{Minute } (^m): \frac{1}{60} \text{ } ^h = \frac{1}{1440} \text{ circle} = 15'$$

$$\text{Second } (^s): \frac{1}{60} \text{ } ^m = \frac{1}{3600} \text{ } ^h = \frac{1}{86400} \text{ circle} = 15''$$

Declination (δ)

Declination is measured using normal degrees (see *Degrees of Angle*) from the equator.

B.6 United States customary units (aka Imperial Units)

B.6.1 Length

$$\text{Point } (p): = \frac{127}{360} \text{ mm}$$

$$\text{Pica } (P/): = 12 p = \frac{127}{30} \text{ mm}$$

$$\text{Inch } (in \text{ or } \text{")}: = 6 P/ = 25.4 \text{ mm}$$

$$\text{Foot } (ft \text{ or } \text{'}): = 12 in = 0.3048 \text{ m}$$

$$\text{Yard } (yd): = 3 ft = 0.9144 \text{ m}$$

$$\text{Mile } (Mi): = 5280 ft = 1760 yd = 1.609344 \text{ km}$$

B.7 Degrees of Angle

$$\text{Degree } (^\circ): \frac{1}{360} \text{ circle} = \frac{\pi}{180} \text{ rad} \approx 0.0174532925199433 \text{ rad}$$

$$\text{Minute of arc } (arcmin \text{ or } \text{'}): \frac{1}{60} \text{ } ^\circ = \frac{1}{21600} \text{ circle} = \frac{\pi}{10800} \text{ rad}$$

$$\text{Second of arc } (arcsec \text{ or } \text{")}: \frac{1}{60} \text{ } arcmin = \frac{1}{3600} \text{ } ^\circ = \frac{1}{206265} \text{ circle} = \frac{\pi}{648000} \text{ rad}$$

B.8 Miscellaneous Units

Area: barn

$$\bullet = 100 fm^2 = 1 \times 10^{-29} m^2$$

B.8.1 Pressure

$$\text{Bar } (bar): = 10^5 Pa \approx 0.9869 \text{ atm}$$

$$\text{Atmosphere } (atm): = 101325 Pa = 1.01325 \text{ bar}$$

$$\text{Torr } (torr): = \frac{1}{760} \text{ atm} = \frac{101325}{760} Pa \approx 133.3224 Pa$$

B.9 Prefixes

atto (a) = $\times 10^{-18}$

femto (f) = $\times 10^{-15}$

pico (p) = $\times 10^{-12}$

nano (n) = $\times 10^{-9}$

micro (μ) = $\times 10^{-6}$

milli (m) = $\times 10^{-3}$

centi (c) = $\times 10^{-2}$

deca (da) = $\times 10^1$

hecto (h) = $\times 10^2$

kilo (k) = $\times 10^3$

mega (M) = $\times 10^6$

giga (G) = $\times 10^9$

tera (T) = $\times 10^{12}$

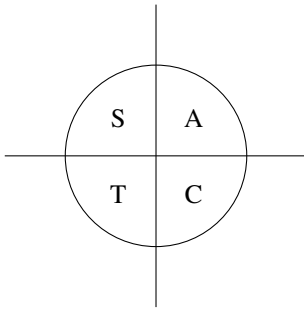
peta (P) = $\times 10^{15}$

exa (E) = $\times 10^{18}$

Appendix C

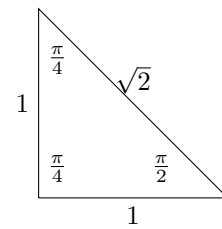
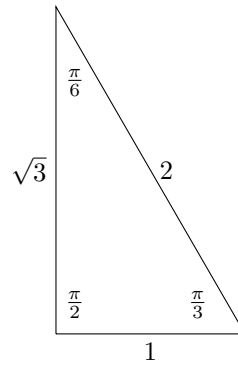
Mathematical Stuff

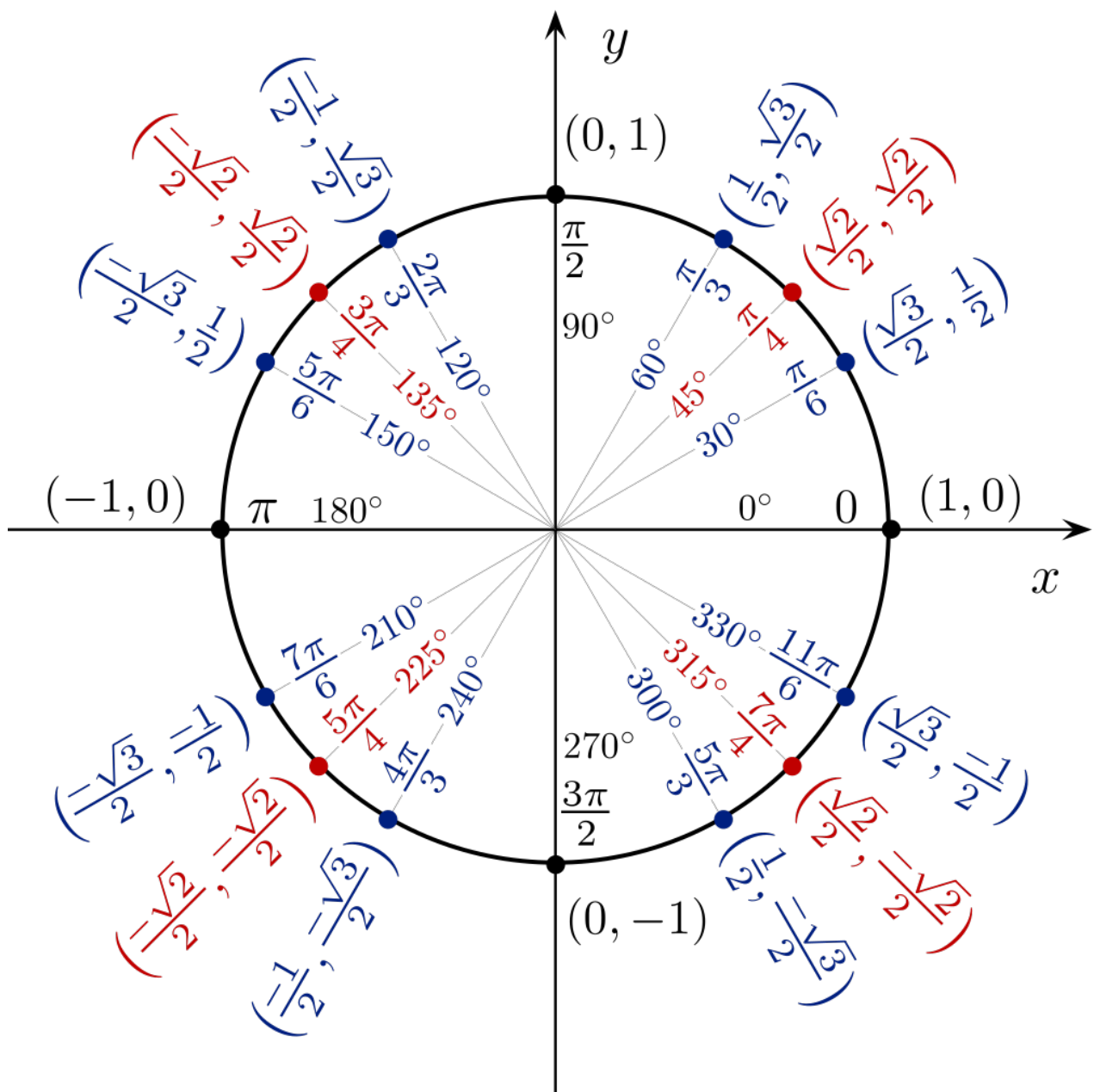
C.1 Trigonometric Values



rad	°	sin	cos	tan
0	0	0	1	0
$\pi/6$	30	1/2	$\sqrt{3}/2$	$1/\sqrt{3}$
$\pi/4$	45	$1/\sqrt{2}$	$1/\sqrt{2}$	1
$\pi/3$	60	$\sqrt{3}/2$	1/2	$\sqrt{3}$
$\pi/2$	90	1	0	$\pm\infty$
$2\pi/3$	120	$\sqrt{3}/2$	-1/2	$-\sqrt{3}$
$3\pi/4$	135	$1/\sqrt{2}$	$-1/\sqrt{2}$	-1
$5\pi/6$	150	1/2	$-\sqrt{3}/2$	$-1/\sqrt{3}$
π	180	0	-1	0
$7\pi/6$	210	-1/2	$\sqrt{3}/2$	$1/\sqrt{3}$
$5\pi/4$	225	$-1/\sqrt{2}$	$-1/\sqrt{2}$	1
$4\pi/3$	240	$-\sqrt{3}/2$	-1/2	$\sqrt{3}$
$3\pi/2$	270	-1	0	$\pm\infty$
$5\pi/3$	300	$-\sqrt{3}/2$	1/2	$-\sqrt{3}$
$7\pi/4$	315	$-1/\sqrt{2}$	$1/\sqrt{2}$	-1
$11\pi/6$	330	-1/2	$\sqrt{3}/2$	$-1/\sqrt{3}$
2π	360	0	1	0

C.1.1 Pythagorean Triples



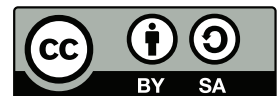


Appendix D

Boring stuff

D.1 Licensing

This work is licensed under a [Creative Commons “Attribution-ShareAlike 4.0 International”](https://creativecommons.org/licenses/by-sa/4.0/) license.



(Basically, as long as you credit me and share under a similar license, feel free to use this however you want)

D.2 Contact

Visit www.webofworlds.net for science fiction, science fact, geeky opinions, and maybe some Python code. Suggestions or corrections are welcome at webofworlds@gmail.com

D.3 Source Code

Available here:

<https://github.com/Lachimax/The-Ultimate-Cheat-Sheet-for-Astrophysics-Students>

D.4 Version History

- v **0.1 2016**: This project is begun in a trio of physical exercise books as *The Little Book of Physics Formulae*, *The Little Book of Mathematics Formulae*, and *The Little Book of Astronomy Formulae*
- v **0.6 2016**: The process of transferring the formulae from paper to LaTeX is initiated, but abandoned (or drifted away from).
- v **0.7 2018-03-20**: The project is resurrected (probably because the author started MRes), uploaded to Overleaf, and cleaned up.
- v **0.8 2018-07-12**: Remaining formulae imported from the original books.
- v **0.9 2018-07-26**: Further formulae imported from undergrad formula sheets.
- v **1.0 2018-07-28**: First public release, with some additions from 0.9.
- v **1.0.1 2018-07-31**: Minor corrections, added “dynamical timescale” (2.2.2) and some more formulae to the Statistics chapter (it was looking a little bare).
- v **1.1 2019-11-29**: Font change; added Planck units and some other miscellaneous units to Units of Measurement; fine structure constant to Physical Constants (why wasn’t it already there?); stellar luminosity; formulae for Green’s functions and other differential equation techniques; minor corrections; Gaussian distributions to Statistics section; more detailed unit circle.
- v **1.1.1**: Cosmological equations; minor typographical corrections (correct differential operator and bracket sizes); added pixel scale equation; using siunitx for units now.

D.5 Credits

Unit Circle: By Jim.belk [CC BY-SA 3.0 (<https://creativecommons.org/licenses/by-sa/3.0>) or GFDL (<http://www.gnu.org/copyleft/fdl.html>)], from Wikimedia Commons (https://commons.wikimedia.org/wiki/File:Unit_circle_angles_color.svg)

Periodic Table: By Dmarcus100 [CC BY-SA 4.0 (<https://creativecommons.org/licenses/by-sa/4.0>)], from Wikimedia Commons (https://commons.wikimedia.org/wiki/File:Periodic_Table_Of_Elements_Atomic_Mass_Black_And_White.jpg)

1 H Hydrogen 1.008	4 Be Beryllium 9.012	10 Ne Neon 20.180	2 He Helium 4.003
3 Li Lithium 6.94	12 Mg Magnesium 24.305	18 Ar Argon 39.948	5 B Boron 10.81
11 Na Sodium 22.990	20 Ca Calcium 40.078	17 Cl Chlorine 35.45	6 C Carbon 12.011
19 K Potassium 39.098	38 Sr Strontium 87.62	35 Br Bromine 79.904	7 N Nitrogen 14.007
37 Rb Rubidium 85.468	88 Ra Radium [226]	81 Tl Thallium 204.38	8 O Oxygen 15.999
55 Cs Cesium 132.905	89 - 102 **	82 Pb Lead 207.2	9 F Fluorine 18.998
71 Lu Lutetium 174.967	103 Lr Lawrencium [262]	83 Bi Bismuth 208.980	10 No Nobelium [259]
72 Hf Hafnium 178.49	104 Rf Rutherfordium [261]	84 Po Polonium [209]	11 Mc Moscovium [289]
73 Ta Tantalum 180.948	105 Db Dubnium [270]	85 At Astatine [210]	115 Mc Moscovium [289]
74 W Tungsten 183.84	106 Sg Seaborgium [269]	86 Rn Radon [222]	116 Lv Livermorium [293]
75 Re Rhenium 186.207	107 Bh Bohrium [270]	87 Fr Francium [223]	117 Ts Tennessine [293]
76 Os Osmium 190.23	108 Hs Hassium [270]	88 Ra Radium [226]	118 Og Oganesson [294]
77 Ir Iridium 192.217	109 Mt Meitnerium [278]	89 - 102 **	
78 Rh Rhodium 102.906	110 Ds Darmstadtium [281]	91 Pa Protactinium 231.036	
79 Pt Platinum 195.084	111 Rg Roentgenium [281]	92 U Uranium 238.029	
80 Au Gold 196.967	112 Cn Copernicium [285]	93 Np Neptunium [237]	
81 Hg Mercury 200.592	113 Nh Nihonium [286]	94 Pu Plutonium [244]	
82 Pb Lead 207.2	114 Fl Flerovium [289]	95 Am Americium [243]	
83 Bi Bismuth 208.980	115 Mc Moscovium [289]	96 Cm Curium [247]	
84 Po Polonium [209]	116 Lv Livermorium [293]	97 Bk Berkelium [247]	
85 At Astatine [210]	117 Ts Tennessine [293]	98 Cf Californium [251]	
86 Rn Radon [222]	118 Og Oganesson [294]	99 Es Einsteinium [252]	
87 Fr Francium [223]		100 Fm Fermium [257]	
		101 Md Mendelevium [258]	
		102 No Nobelium [259]	
		103 Lr Lawrencium [262]	
		104 Rf Rutherfordium [261]	
		105 Db Dubnium [270]	
		106 Sg Seaborgium [269]	
		107 Bh Bohrium [270]	
		108 Hs Hassium [270]	
		109 Mt Meitnerium [278]	
		110 Ds Darmstadtium [281]	
		111 Rg Roentgenium [281]	
		112 Cn Copernicium [285]	
		113 Nh Nihonium [286]	
		114 Fl Flerovium [289]	
		115 Mc Moscovium [289]	
		116 Lv Livermorium [293]	
		117 Ts Tennessine [293]	
		118 Og Oganesson [294]	
		119 Uu Ununennium [295]	
		120 Uub Unbibium [296]	
		121 Uut Untrium [297]	
		122 Uuq Unquadrium [298]	
		123 Uuq Unquadrium [298]	
		124 Uup Unpentium [299]	
		125 Uuq Unquadrium [298]	
		126 Uuq Unquadrium [298]	
		127 Uuq Unquadrium [298]	
		128 Uuq Unquadrium [298]	
		129 Uuq Unquadrium [298]	
		130 Uuq Unquadrium [298]	
		131 Uuq Unquadrium [298]	
		132 Uuq Unquadrium [298]	
		133 Uuq Unquadrium [298]	
		134 Uuq Unquadrium [298]	
		135 Uuq Unquadrium [298]	
		136 Uuq Unquadrium [298]	
		137 Uuq Unquadrium [298]	
		138 Uuq Unquadrium [298]	
		139 Uuq Unquadrium [298]	
		140 Uuq Unquadrium [298]	
		141 Uuq Unquadrium [298]	
		142 Uuq Unquadrium [298]	
		143 Uuq Unquadrium [298]	
		144 Uuq Unquadrium [298]	
		145 Uuq Unquadrium [298]	
		146 Uuq Unquadrium [298]	
		147 Uuq Unquadrium [298]	
		148 Uuq Unquadrium [298]	
		149 Uuq Unquadrium [298]	
		150 Uuq Unquadrium [298]	
		151 Uuq Unquadrium [298]	
		152 Uuq Unquadrium [298]	
		153 Uuq Unquadrium [298]	
		154 Uuq Unquadrium [298]	
		155 Uuq Unquadrium [298]	
		156 Uuq Unquadrium [298]	
		157 Uuq Unquadrium [298]	
		158 Uuq Unquadrium [298]	
		159 Uuq Unquadrium [298]	
		160 Uuq Unquadrium [298]	
		161 Uuq Unquadrium [298]	
		162 Uuq Unquadrium [298]	
		163 Uuq Unquadrium [298]	
		164 Uuq Unquadrium [298]	
		165 Uuq Unquadrium [298]	
		166 Uuq Unquadrium [298]	
		167 Uuq Unquadrium [298]	
		168 Uuq Unquadrium [298]	
		169 Uuq Unquadrium [298]	
		170 Uuq Unquadrium [298]	
		171 Uuq Unquadrium [298]	
		172 Uuq Unquadrium [298]	
		173 Uuq Unquadrium [298]	
		174 Uuq Unquadrium [298]	
		175 Uuq Unquadrium [298]	
		176 Uuq Unquadrium [298]	
		177 Uuq Unquadrium [298]	
		178 Uuq Unquadrium [298]	
		179 Uuq Unquadrium [298]	
		180 Uuq Unquadrium [298]	
		181 Uuq Unquadrium [298]	
		182 Uuq Unquadrium [298]	
		183 Uuq Unquadrium [298]	
		184 Uuq Unquadrium [298]	
		185 Uuq Unquadrium [298]	
		186 Uuq Unquadrium [298]	
		187 Uuq Unquadrium [298]	
		188 Uuq Unquadrium [298]	
		189 Uuq Unquadrium [298]	
		190 Uuq Unquadrium [298]	
		191 Uuq Unquadrium [298]	
		192 Uuq Unquadrium [298]	
		193 Uuq Unquadrium [298]	
		194 Uuq Unquadrium [298]	
		195 Uuq Unquadrium [298]	
		196 Uuq Unquadrium [298]	
		197 Uuq Unquadrium [298]	
		198 Uuq Unquadrium [298]	
		199 Uuq Unquadrium [298]	
		200 Uuq Unquadrium [298]	
		201 Uuq Unquadrium [298]	
		202 Uuq Unquadrium [298]	
		203 Uuq Unquadrium [298]	
		204 Uuq Unquadrium [298]	
		205 Uuq Unquadrium [298]	
		206 Uuq Unquadrium [298]	
		207 Uuq Unquadrium [298]	
		208 Uuq Unquadrium [298]	
		209 Uuq Unquadrium [298]	
		210 Uuq Unquadrium [298]	
		211 Uuq Unquadrium [298]	
		212 Uuq Unquadrium [298]	
		213 Uuq Unquadrium [298]	
		214 Uuq Unquadrium [298]	
		215 Uuq Unquadrium [298]	
		216 Uuq Unquadrium [298]	
		217 Uuq Unquadrium [298]	
		218 Uuq Unquadrium [298]	
		219 Uuq Unquadrium [298]	
		220 Uuq Unquadrium [298]	
		221 Uuq Unquadrium [298]	
		222 Uuq Unquadrium [298]	
		223 Uuq Unquadrium [298]	
		224 Uuq Unquadrium [298]	
		225 Uuq Unquadrium [298]	
		226 Uuq Unquadrium [298]	
		227 Uuq Unquadrium [298]	
		228 Uuq Unquadrium [298]	
		229 Uuq Unquadrium [298]	
		230 Uuq Unquadrium [298]	
		231 Uuq Unquadrium [298]	
		232 Uuq Unquadrium [298]	
		233 Uuq Unquadrium [298]	
		234 Uuq Unquadrium [298]	
		235 Uuq Unquadrium [298]	
		236 Uuq Unquadrium [298]	
		237 Uuq Unquadrium [298]	
		238 Uuq Unquadrium [298]	
		239 Uuq Unquadrium [298]	
		240 Uuq Unquadrium [298]	
		241 Uuq Unquadrium [298]	
		242 Uuq Unquadrium [298]	
		243 Uuq Unquadrium [298]	
		244 Uuq Unquadrium [298]	
		245 Uuq Unquadrium [298]	
		246 Uuq Unquadrium [298]	
		247 Uuq Unquadrium [298]	
		248 Uuq Unquadrium [298]	
		249 Uuq Unquadrium [298]	
		250 Uuq Unquadrium [298]	
		251 Uuq Unquadrium [298]	
		252 Uuq Unquadrium [298]	
		253 Uuq Unquadrium [298]	
		254 Uuq Unquadrium [298]	
		255 Uuq Unquadrium [298]	
		256 Uuq Unquadrium [298]	
		257 Uuq Unquadrium [298]	
		258 Uuq Unquadrium [298]	
		259 Uuq Unquadrium [298]	
		260 Uuq Unquadrium [298]	
		261 Uuq Unquadrium [298]	
		262 Uuq Unquadrium [298]	
		263 Uuq Unquadrium [298]	
		264 Uuq Unquadrium [298]	
		265 Uuq Unquadrium [298]	
		266 Uuq Unquadrium [298]	
		267 Uuq Unquadrium [298]	
		268 Uuq Unquadrium [298]	
		269 Uuq Unquadrium [298]	
		270 Uuq Unquadrium [298]	
		271 Uuq Unquadrium [298]	
		272 Uuq Unquadrium [298]	
		273 Uuq Unquadrium [298]	
		274 Uuq Unquadrium [298]	
		275 Uuq Unquadrium [298]	
		276 Uuq Unquadrium [298]	
		277 Uuq Unquadrium [298]	
		278 Uuq Unquadrium [298]	
		279 Uuq Unquadrium [298]	
		280 Uuq Unquadrium [298]	
		281 Uuq Unquadrium [298]	
		282 Uuq Unquadrium [298]	
		283 Uuq Unquadrium [298]	
		284 Uuq Unquadrium [298]	
		285 Uuq Unquadrium [298]	
		286 Uuq Unquadrium [298]	
		287 Uuq Unquadrium [298]	
		288 Uuq Unquadrium [298]	
		289 Uuq Unquadrium [298]	
		290 Uuq Unquadrium [298]	
		291 Uuq Unquadrium [298]	
		29	